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在 time scales 上的衝擊動態方程的週期邊界值問題
研究成果報告(精簡版)

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1 Introduction

Maximum principles are an important tool in the study of partial differential and difference equations. For example, they can be used to obtain the existence and uniqueness of solutions and to approximate it. Consequently the theory of maximum principles in difference and differential equations has been investigated extensively, see for example [1] and [2] and the references cited therein.

In recent years, the study of dynamic equations on time scales has received a lot of attentions since it not only can unify the calculation of difference and differential equations but also has various applications. In particular, the maximum principles have been established in [4] for the second order ordinary dynamic operator and [5] for the elliptic dynamic operator. Motivated by the above work, in this paper, we study the maximum principles for the elliptic dynamic operator

$$\mathcal{L}[u] := \sum_{i=1}^{n}(u^{\nabla_i\Delta_i} + B_i u^{\Delta_i} + C_i u^{\nabla_i})$$

and the parabolic dynamic operator

$$L[u] := \sum_{i=1}^{n}(u^{\nabla_i\Delta_i} + \tilde{B}_i u^{\Delta_i} + \tilde{C}_i u^{\nabla_i}) - u^{\nabla_{n+1}}.$$

Our results improve the results in [5].

This paper is organized as follows. Section 2 contains some basic definitions and the necessary results about time scales. In Section 3, we present the maximum principles for the elliptic dynamic operators. Finally, in section 4, we establish the maximum principles for parabolic operators, and apply it to obtain some useful applications.

2 Preliminary

For completeness, we state some fundamental definitions and results concerning partial dynamic equations on time scales that we will use in the sequel. It can be regarded as a generalization of the one-dimensional case. More details can be found in [6], [7], [8], and [9].

A time scale is an arbitrary nonempty closed subset of $\mathbb{R}$. Throughout this paper, we denote $I = \{1, 2, \cdots, n\}$, where $n \in \mathbb{N}$, and we assume that $\mathbb{T}_i$, for each $i \in I$, is a time scale and the set

$$\Lambda = \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n = \{t = (t_1, t_2, \cdots, t_n) \mid t_i \in \mathbb{T}_i \text{ for each } i \in I\},$$

defined by the Cartesian product is an n-dimensional time scale.

Definition 2.1 *For each $i \in I$, the mappings σ_i, $\rho_i : \mathbb{T}_i \to \mathbb{T}_i$ defined by*

$$\sigma_i(u) := \begin{cases} \inf\{v \in \mathbb{T}_i \mid v > u\}, & if\ \ u \neq \max \mathbb{T}_i, \\ \max \mathbb{T}_i, & if\ \ u = \max \mathbb{T}_i, \end{cases}$$

and

$$\rho_i(u) := \begin{cases} \sup\{v \in \mathbb{T}_i \mid v < u\}, & if\ \ u \neq \min \mathbb{T}_i, \\ \min \mathbb{T}_i, & if\ \ u = \min \mathbb{T}_i, \end{cases}$$

are called the ith forward and backward jump operators respectively. In this definition, the corresponding graininess functions μ_i, $\nu_i : \mathbb{T}_i \to [0, \infty)$ are defined by

$$\mu_i(u) := \sigma_i(u) - u, \qquad \nu_i(u) := u - \rho_i(u).$$

For convenience, we define the functions $\hat{\sigma}_i$, $\hat{\rho}_i : \Lambda \to \Lambda$ by

$$\hat{\sigma}_i(t) = (t_1, t_2, \cdots, t_{i-1}, \sigma_i(t_i), t_{i+1}, \cdots, t_n),$$

and

$$\hat{\rho}_i(t) = (t_1, t_2, \cdots, t_{i-1}, \rho_i(t_i), t_{i+1}, \cdots, t_n),$$

for any $t \in \Lambda$ *and* $i \in I$. *In addition, if* $u : \Lambda \to \mathbb{R}$ *is a function, then the functions* $u^{\hat{\sigma}_i}$, $u^{\hat{\rho}_i} : \Lambda \to \mathbb{R}$ *are defined by*

$$u^{\hat{\sigma}_i}(t) = u(\hat{\sigma}_i(t)) \qquad \text{and} \qquad u^{\hat{\rho}_i}(t) = u(\hat{\rho}_i(t)),$$

for any $t \in \Lambda$ *and* $i \in I$.

Definition 2.2 *A point* t *in* Λ *is said to be* i*-right dense if* $t_i < \max \mathbb{T}_i$ *and* $\sigma_i(t_i) = t_i$, *and* i*-left dense if* $t_i > \min \mathbb{T}_i$ *and* $\rho_i(t_i) = t_i$. *Also, if* $\sigma_i(t_i) > t_i$ *then* t *is called* i*-right scattered, and if* $\rho_i(t_i) < t_i$ *then* t *is called* i*-left scattered. Moreover, we say that* t *is* i*-scattered if it is both* i*-left scattered and* i*-right scattered, and* i*-dense if it is both* i*-left dense and* i*-right dense.*

Definition 2.3 *For each* $i \in I$, *let*

$$(\mathbb{T}_i)^{\mathcal{K}} = \begin{cases} \mathbb{T}_i \backslash \max \mathbb{T}_i, & \text{if } \mathbb{T}_i \text{ has a left scattered maximum}, \\ \mathbb{T}_i, & \text{if } \mathbb{T}_i \text{ has a left dense maximum}. \end{cases}$$

Then we can define

$$\Lambda^{\mathcal{K}} = (\mathbb{T}_1)^{\mathcal{K}} \times (\mathbb{T}_2)^{\mathcal{K}} \times \cdots \times (\mathbb{T}_n)^{\mathcal{K}}.$$

Assume $u : \Lambda \to \mathbb{R}$ *is a function and let* $t \in \Lambda^{\mathcal{K}}$. *Then we define* $u^{\Delta_i}(t)$ *to be the number (provided it exists) with the property that given any* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that*

$$\mid [u(\hat{\sigma}_i(t)) - u(s)] - u^{\Delta_i}(t)[\hat{\sigma}_i(t) - s] \mid \leq \varepsilon \mid \hat{\sigma}_i(t) - s \mid,$$

for all $s \in (t - \delta e_i, t + \delta e_i) \cap \Lambda$, *where* $\{e_i \mid i \in I\}$ *denotes the natural basis for* $\mathbb{R}^n$. *In this case, we call* $u^{\Delta_i}(t)$ *the partial delta derivative of* u *at* t *with respect to* t_i.

In particular, if we choose $n = 1$ *in this definition, then* u *is a single variable* *function from* $\mathbb{T}_1$ *into* $\mathbb{R}$*, and we denote the delta derivative of* u *at* $t \in (\mathbb{T}_1)^{\mathcal{K}}$ *by* $u^{\Delta}(t)$*.*

Definition 2.4 *For each* $i \in I$*, let*

$$(\mathbb{T}_i)_{\mathcal{K}} = \begin{cases} \mathbb{T}_i \backslash \min \mathbb{T}_i, & \text{if } \mathbb{T}_i \text{ has a right scattered minimum,} \\ \mathbb{T}_i, & \text{if } \mathbb{T}_i \text{ has a right dense minimum.} \end{cases}$$

Then we can define

$$\Lambda_{\mathcal{K}} = (\mathbb{T}_1)_{\mathcal{K}} \times (\mathbb{T}_2)_{\mathcal{K}} \times \cdots \times (\mathbb{T}_n)_{\mathcal{K}}.$$

Assume $u : \Lambda \to \mathbb{R}$ *is a function and let* $t \in \Lambda_{\mathcal{K}}$*. Then we define* $u^{\nabla_i}(t)$ *to be the number (provided it exists) with the property that given any* $\varepsilon > 0$*, there exists a* $\delta > 0$ *such that*

$$\mid [u(\hat{\rho}_i(t)) - u(s)] - u^{\nabla_i}(t)[\hat{\rho}_i(t) - s] \mid \leq \varepsilon \mid \hat{\rho}_i(t) - s \mid,$$

for all $s \in (t - \delta e_i, t + \delta e_i) \cap \Lambda$*. In this case, we call* $u^{\nabla_i}(t)$ *the partial nabla derivative of* u *at* t *with respect to* t_i*.*
In particular, if we choose $n = 1$ *in this definition, then* u *is a single variable function from* $\mathbb{T}_1$ *into* $\mathbb{R}$*, and we denote the nabla derivative of* u *at* $t \in (\mathbb{T}_1)_{\mathcal{K}}$ *by* $u^{\nabla}(t)$*.*

For convenience, we denote the intersection of $\Lambda^{\mathcal{K}}$ and $\Lambda_{\mathcal{K}}$ by $\Lambda^{\mathcal{K}}_{\mathcal{K}}$, i.e.,

$$\Lambda^{\mathcal{K}}_{\mathcal{K}} = (\mathbb{T}_1)^{\mathcal{K}}_{\mathcal{K}} \times (\mathbb{T}_2)^{\mathcal{K}}_{\mathcal{K}} \times \cdots \times (\mathbb{T}_n)^{\mathcal{K}}_{\mathcal{K}}.$$

Definition 2.5 *Let the functions* U*,* $u : \Lambda \to \mathbb{R}$ *satisfy* $U^{\Delta_i}(t) = u(t)$ *for all* $t \in \Lambda^{\mathcal{K}}$*, then we define* $\int_r^s u(t)\Delta_i t = U(s) - U(r)$ *for all* r*,* $s \in \Lambda$ *and* $i \in I$*. Similarly, we can define* $\int_r^s u(t)\nabla_i t = U(s) - U(r)$ *for all* r*,* $s \in \Lambda$ *if* $U^{\nabla_i}(t) = u(t)$ *for all* $t \in \Lambda_{\mathcal{K}}$ *and* $i \in I$*.*

Definition 2.6 *Let* $\mathbb{T}$ *be an arbitrary time scale, and* $p : \mathbb{T} \to \mathbb{R}$ *be a function* *and satisfy*

$$1 - \nu(t)p(t) \neq 0 \qquad for\ all\ t \in \mathbb{T}_{\mathcal{K}}.$$

Then we define the nabla exponential function by

$$\hat{e}_p(t,s) = exp(\int_s^t g(\tau)\nabla\tau) \qquad for\ s,t \in \mathbb{T},$$

where

$$g(\tau) = \begin{cases} p(\tau), & if\ \nu(\tau) = 0, \\ -\dfrac{1}{\nu(\tau)} Log(1 - \nu(\tau)p(\tau)), & if\ \nu(\tau) \neq 0. \end{cases}$$

Lemma 2.7 *Suppose that α is a negative constant and $s, t, u \in \mathbb{T}$, then*

(a) $\hat{e}_\alpha(t,s) > 0$ *and* $\hat{e}_\alpha(t,t) \equiv 1$;

(b) $\hat{e}_\alpha(t,u)\hat{e}_\alpha(u,s) = \hat{e}_\alpha(t,s)$;

(c) $\hat{e}_\alpha^\nabla(t,s) = \alpha\hat{e}_\alpha(t,s)$.

Lemma 2.8 *Assume that $f : \mathbb{T} \to \mathbb{R}$ is a single variable function and let $t \in \mathbb{T}_{\mathcal{K}}^{\mathcal{K}}$, then we have the following:*

(a) If f is delta or nabla differentiable at t, then f is continuous at t.

(b) If f is continuous at a right-scattered point t, then f is delta differentiable at t with

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(c) If t is right-dense, then f is delta differentiable at t if and only if the limit

$$\lim_{s\to t} \frac{f(t) - f(s)}{t - s}$$

exists. In this case,

$$f^\Delta(t) = \lim_{s\to t} \frac{f(t) - f(s)}{t - s}.$$

(d) If f is delta differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t).$$

(e) If f is continuous at a left-scattered point t, then f is nabla differentiable at t with

$$f^{\nabla}(t) = \frac{f(t) - f(\rho(t))}{\nu(t)}.$$

(f) If t is left-dense, then f is nabla differentiable at t if and only if the limit

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists. In this case,

$$f^{\nabla}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

(g) If f is nabla differentiable at t, then

$$f(\rho(t)) = f(t) - \nu(t) f^{\nabla}(t).$$

Hereafter $[a, b]_{\mathbb{T}}$ represents an interval on time scale $\mathbb{T}$, that is, $[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T}$. Other types of intervals on a time scale can be represented by the similar way.

Lemma 2.9 *Assume that $f : \mathbb{T} \to \mathbb{R}$ is a function, then*

(a) If $f^{\Delta} > 0$ on $[a, b]_{\mathbb{T}}$, then f is strictly increasing on $[a, b]_{\mathbb{T}}$.

(b) If $f > 0$ is a continuous function on $[a, b]_{\mathbb{T}}$, then $\int_a^b f(t)\Delta t > 0$ and $\int_a^b f(t)\nabla t > 0$, where a, $b \in \mathbb{T}$.

Lemma 2.10 *Assume that $f : \mathbb{T} \to \mathbb{R}$ is nabla differentiable and f^{∇} is continuous on $\mathbb{T}_{\mathcal{K}}$. Then f is delta differentiable at t and*

$$f^{\Delta}(t) = f^{\nabla}(\sigma(t)) \qquad \textit{for all } t \in \mathbb{T}^{\mathcal{K}}.$$

3 Maximum principles for elliptic dynamic equations

In this section we first consider the dynamic Laplace operator

$$\Delta_{\mathbb{T}} u := \sum_{i=1}^{n} u^{\nabla_i \Delta_i}.$$

Let

$$\Lambda = [\rho_1(a_1), \sigma_1(b_1)]_{\mathbb{T}_1} \times \cdots \times [\rho_n(a_n), \sigma_n(b_n)]_{\mathbb{T}_n}.$$

We shall study the functions in the set

$$\mathcal{D}(\Lambda) := \{u : \Lambda \to \mathbb{R} \mid u^{\nabla_i \Delta_i} \text{ is continuous in } \Lambda_{\mathcal{K}}^{\mathcal{K}} \text{ for each } i \in I\}.$$

The following lemma provides some basic properties for an interior maximum point of a function in $\mathcal{D}(\Lambda)$.

Lemma 3.1 *Suppose that $u \in \mathcal{D}(\Lambda)$ attains its maximum at an interior point m of Λ. Then, for each $i \in I$, we have*

$$u^{\nabla_i}(m) \geq 0, \qquad u^{\Delta_i}(m) \leq 0, \qquad \textit{and} \qquad u^{\nabla_i \Delta_i}(m) \leq 0.$$

In particular, if m is i-right dense, then

$$u^{\nabla_i}(m) = u^{\Delta_i}(m) = 0.$$

Proof. Since u attains its maximum at an interior point m of Λ, it follows from the definition of u^{∇_i} and u^{Δ_i} that

$$u^{\nabla_i}(m) \geq 0 \qquad \textit{and} \qquad u^{\Delta_i}(m) \leq 0, \tag{1}$$

for each $i \in I$. Let us divide our proof into two cases according to the point type of m with respect to the ith component.

(i) m is i-right dense:

In this case, by applying Lemma 2.10, we have that

$$u^{\Delta_i}(m) = u^{\nabla_i}(\hat{\sigma}_i(m)) = u^{\nabla_i}(m),$$

and consequently, together with (1), we conclude that

$$u^{\nabla_i}(m) = u^{\Delta_i}(m) = 0.$$

Now we want to show that $u^{\nabla_i \Delta_i}(m) \leq 0$. For contradiction, we assume that $u^{\nabla_i \Delta_i}(m) > 0$. Then the continuity of $u^{\nabla_i \Delta_i}$ and Lemma 2.9 imply that there exists a $\delta > 0$ such that u^{∇_i} is strictly increasing in t_i on J, where J denotes the set of all points $t \in \Lambda$ lying on the line segment joining m and $m + \delta e_i$. Since m is i-right dense, without loss of generality, we may assume that $m_i + \delta \in \mathbb{T}_i$. Since $u^{\nabla_i}(m) = 0$, it follows that $u^{\nabla_i}(t) > 0$ for all $t \in J$. Then, by applying Lemma 2.9, we easily get

$$\int_m^{m+\delta e_i} u^{\nabla_i}(s) \nabla_i s = u(m + \delta e_i) - u(m) > 0,$$

which contradicts the fact that $u(m)$ is the maximum value on Λ.

(ii) m is i-right scattered.

Note that

$$u^{\nabla_i}(\hat{\sigma}_i(m)) = \frac{u(\hat{\sigma}_i(m)) - u(\hat{\rho}_i(\hat{\sigma}_i(m)))}{\sigma_i(m_i) - \rho_i(\sigma_i(m_i))} = \frac{u(\hat{\sigma}_i(m)) - u(m)}{\sigma_i(m_i) - m_i} = u^{\Delta_i}(m).$$

Together with (1), we obtain

$$u^{\nabla_i \Delta_i}(m) = \frac{u^{\nabla_i}(\hat{\sigma}_i(m)) - u^{\nabla_i}(m)}{\sigma_i(m_i) - m_i} = \frac{u^{\Delta_i}(m) - u^{\nabla_i}(m)}{\sigma_i(m_i) - m_i} \leq 0.$$

□

Corollary 3.2 *If $u \in \mathcal{D}(\Lambda)$ satisfies*

$$\Delta_{\mathbb{T}} u > 0, \qquad \textit{in } \Lambda_{\mathcal{K}}^{\mathcal{K}}, \tag{2}$$

then u cannot attain its maximum at an interior point of Λ.

Proof. For contradiction, we assume that u attains its maximum at an interior point m of Λ. By applying Lemma 3.1, we have that $u^{\nabla_i \Delta_i}(m) \leq 0$ for each

$i \in I$. This implies that

$$\Delta_{\mathbb{T}} u(m) = \sum_{i=1}^{n} u^{\nabla_i \Delta_i}(m) \le 0,$$

which contradicts (2). $\square$

Next we consider the more general operator which contains the first-derivative terms

$$\mathcal{L}[u] := \sum_{i=1}^{n} (u^{\nabla_i \Delta_i} + B_i u^{\Delta_i} + C_i u^{\nabla_i}) = \Delta_{\mathbb{T}} u + \sum_{i=1}^{n} (B_i u^{\Delta_i} + C_i u^{\nabla_i}).$$

Following the statement of Lemma 3.1, for each $t \in \Lambda$, we define the auxiliary index sets

$$I^t_{RD} := \{i \in I : t_i = \sigma_i(t_i)\},$$
$$I^t_{RS} := \{i \in I : t_i < \sigma_i(t_i)\}.$$

Corollary 3.3 *If $u \in \mathcal{D}(\Lambda)$ satisfies*

$$\mathcal{L}[u] > 0, \qquad \text{in } \Lambda^{\mathcal{K}}_{\mathcal{K}}, \tag{3}$$

and let B_i and C_i satisfy

$$\begin{cases} B_i(t) \ge 0, \\ \\ C_i(t) \le 0, \end{cases} \tag{4}$$

for each $t \in \Lambda^{\mathcal{K}}_{\mathcal{K}}$ which is i-right scattered and $i \in I$. Then u cannot attain its maximum at an interior point of Λ.

Proof. For contradiction, we assume that u attains its maximum at an interior point m of Λ. Lemma 3.1 yields that at the point m, we have

$$u^{\Delta_i}(m) = 0,\ u^{\nabla_i}(m) = 0, \text{ and } u^{\nabla_i \Delta_i}(m) \le 0 \qquad \text{if } i \in I^m_{RD},$$
$$u^{\Delta_i}(m) \le 0,\ u^{\nabla_i}(m) \ge 0, \text{ and } u^{\nabla_i \Delta_i}(m) \le 0 \qquad \text{if } i \in I^m_{RS}.$$

Therefore, together with the assumption (4), we have that

$$\begin{aligned}
&\mathcal{L}[u](m)\\
&= \sum_{i=1}^{n}(u^{\nabla_i\Delta_i}(m) + B_i(m)u^{\Delta_i}(m) + C_i(m)u^{\nabla_i}(m))\\
&= \sum_{i\in I_{RD}^m} u^{\nabla_i\Delta_i}(m) + \sum_{i\in I_{RS}^m}(u^{\nabla_i\Delta_i}(m) + B_i(m)u^{\Delta_i}(m) + C_i(m)u^{\nabla_i}(m))\\
&\leq 0,
\end{aligned}$$

which contradicts (3). □

Theorem 3.4 *Let $u \in \mathcal{D}(\Lambda)$ satisfy the inequality (3) and let B_i and C_i satisfy*

$$\begin{cases} 1 + B_i(t)\mu_i(t_i) \geq 0, \\ -1 + C_i(t)\mu_i(t_i) \leq 0, \end{cases} \tag{5}$$

for each $t \in \Lambda_{\mathcal{K}}^{\mathcal{K}}$ which is i-right scattered and $i \in I$. Then u cannot attain its maximum at an interior point of Λ.

Proof. For contradiction, we assume that u attains its maximum at an interior point m of Λ. Then, by applying Lemma 3.1, we can rewrite $\mathcal{L}[u](m)$ in the following way:

$$\begin{aligned}
&\mathcal{L}[u](m)\\
&= \sum_{i=1}^{n}(u^{\nabla_i\Delta_i}(m) + B_i(m)u^{\Delta_i}(m) + C_i(m)u^{\nabla_i}(m))\\
&= \sum_{i\in I_{RD}^m} u^{\nabla_i\Delta_i}(m) + \sum_{i\in I_{RS}^m}(u^{\nabla_i\Delta_i}(m) + B_i(m)u^{\Delta_i}(m) + C_i(m)u^{\nabla_i}(m))\\
&= \sum_{i\in I_{RD}^m} u^{\nabla_i\Delta_i}(m) + \sum_{i\in I_{RS}^m}(\frac{u^{\Delta_i}(m) - u^{\nabla_i}(m)}{\mu_i(m_i)} + B_i(m)u^{\Delta_i}(m) + C_i(m)u^{\nabla_i}(m)).
\end{aligned} \tag{6}$$

If $I = I_{RD}^m$, then (6) implies that

$$\mathcal{L}[u](m) = \sum_{i\in I_{RD}^m} u^{\nabla_i\Delta_i}(m) \leq 0,$$

which contradicts (3). Otherwise, let us define the auxiliary functions

$$\hat{\mu}(t) := \prod_{j \in I_{RS}^t} \mu_j(t_j), \qquad \hat{\mu}_{-i}(t) := \prod_{\substack{j \in I_{RS}^t \\ j \neq i}} \mu_j(t_j).$$

Obviously, if $i \in I_{RS}^t$ we have

$$\hat{\mu}(t) = \hat{\mu}_{-i}(t)\mu_i(t_i). \tag{7}$$

We multiply both sides of the equality (6) by $\hat{\mu}(m) > 0$ and use (7) to obtain

$$\begin{aligned}
&\hat{\mu}(m)\mathcal{L}[u](m) \\
&= \hat{\mu}(m) \sum_{i \in I_{RD}^m} u^{\nabla_i \Delta_i}(m) \\
&\quad + \hat{\mu}_{-i}(m)\mu_i(m_i) \sum_{i \in I_{RS}^m} \Big(\frac{u^{\Delta_i}(m) - u^{\nabla_i}(m)}{\mu_i(m_i)} + B_i(m)u^{\Delta_i}(m) + C_i(m)u^{\nabla_i}(m)\Big) \\
&= \hat{\mu}(m) \sum_{i \in I_{RD}^m} u^{\nabla_i \Delta_i}(m) \\
&\quad + \hat{\mu}_{-i}(m) \sum_{i \in I_{RS}^m} [(1 + B_i(m)\mu_i(m_i))u^{\Delta_i}(m) + (-1 + C_i(m)\mu_i(m_i))u^{\nabla_i}(m)].
\end{aligned}$$

Lemma 3.1 together with the assumptions (5), and positivity of $\hat{\mu}(m)$ and $\hat{\mu}_{-i}(m)$ imply that

$$\hat{\mu}(m)\mathcal{L}[u](m) \leq 0,$$

which contradicts (3). Therefore we conclude that u cannot achieve its maximum at an interior point of Λ. $\square$

4 Maximum principles for parabolic dynamic equations

In this section, we extend our results in the last section to the parabolic dynamic operators. Let Λ be an n-dimensional time scale defined in section 3. Then we define the $(n+1)$-dimensional time scale Ω by

$$\Omega = \Lambda \times [0, T]_{\mathbb{T}_{n+1}},$$

where $\mathbb{T}_{n+1}$ is an arbitrary time scale and $0, T \in \mathbb{T}_{n+1}$. In addition, we set

$$B = \Lambda \times \{0\} \qquad and \qquad S = \partial\Lambda \times (0, T]_{\mathbb{T}_{n+1}},$$

then we can define the parabolic boundary $P\Omega$ by

$$P\Omega = S \cup B.$$

Throughout this section, we study the functions in the set

$$\mathcal{D}(\Omega) := \{u : \Omega \to \mathbb{R} \mid u^{\nabla_i \Delta_i} \text{ is continuous in } \Lambda_{\mathcal{K}}^{\mathcal{K}} \times [0, T]_{\mathbb{T}_{n+1}} \text{ for each } i \in I$$
$$\text{and } u^{\nabla_{n+1}} \text{ is continuous in } \Lambda \times ([0, T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}\}.$$

Corollary 4.1 *If $u \in \mathcal{D}(\Omega)$ satisfies*

$$\Delta_{\mathbb{T}} u - u^{\nabla_{n+1}} = \sum_{i=1}^{n} u^{\nabla_i \Delta_i} - u^{\nabla_{n+1}} > 0, \qquad in\ \Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0, T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}, \qquad (8)$$

Then u cannot attain its maximum anywhere other than on the parabolic boundary.

Proof. For contradiction, we assume that u attains its maximum at a point $m \in \Omega \setminus P\Omega$. This implies that $m \in \Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0, T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}$. Therefore, by applying Lemma 3.1, we have

$$u^{\nabla_i \Delta_i}(m) \leq 0 \qquad for\ each\ i \in I.$$

Since u attains its maximum at m, by the definition of partial nabla derivative of u, we obtain

$$u^{\nabla_{n+1}}(m) \geq 0. \tag{9}$$

It follows that

$$(\Delta_{\mathbb{T}} u - u^{\nabla_{n+1}})(m) = \sum_{i=1}^{n} u^{\nabla_i \Delta_i}(m) - u^{\nabla_{n+1}}(m) \leq 0,$$

which contradicts (8). $\square$

Similarly, we consider the more general operator

$$L[u] := \sum_{i=1}^{n} (u^{\nabla_i \Delta_i} + \tilde{B}_i u^{\Delta_i} + \tilde{C}_i u^{\nabla_i}) - u^{\nabla_{n+1}}.$$

Corollary 4.2 *If $u \in \mathcal{D}(\Omega)$ satisfies*

$$L[u] > 0, \qquad in\ \Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}, \tag{10}$$

and let $\tilde{B}_i$ and $\tilde{C}_i$ satisfy

$$\begin{cases} \tilde{B}_i(t) \geq 0, \\ \tilde{C}_i(t) \leq 0, \end{cases} \tag{11}$$

for each $t \in \Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}$ which is i-right scattered and $i \in I$. Then u cannot attain its maximum anywhere other than on the parabolic boundary.

Proof. For contradiction, we assume that u attains its maximum at a point $m \in \Omega \setminus P\Omega$. Lemma 3.1 together with the assumptions (11) and (9) imply that

$$\begin{aligned} & L[u](m) \\ = & \sum_{i=1}^{n} (u^{\nabla_i \Delta_i}(m) + \tilde{B}_i(m) u^{\Delta_i}(m) + \tilde{C}_i(m) u^{\nabla_i}(m)) - u^{\nabla_{n+1}}(m) \\ = & \sum_{i \in I_{RD}^m} u^{\nabla_i \Delta_i}(m) + \sum_{i \in I_{RS}^m} (u^{\nabla_i \Delta_i}(m) + \tilde{B}_i(m) u^{\Delta_i}(m) + \tilde{C}_i(m) u^{\nabla_i}(m)) - u^{\nabla_{n+1}}(m) \\ \leq & \ 0, \end{aligned}$$

which contradicts (10). $\square$

Theorem 4.3 *Let $u \in \mathcal{D}(\Omega)$ satisfy the inequality (10) and let $\tilde{B}_i$ and $\tilde{C}_i$ satisfy*

$$\begin{cases} 1 + \tilde{B}_i(t)\mu_i(t_i) \geq 0, \\ -1 + \tilde{C}_i(t)\mu_i(t_i) \leq 0, \end{cases} \tag{12}$$

for each $t \in \Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}$ which is i-right scattered and $i \in I$. Then u cannot attain its maximum anywhere other than on the parabolic boundary.

Proof. For contradiction, we assume that u attains its maximum at a point $m \in \Omega \setminus P\Omega$. As similar as the proof of Theorem 3.4, we rewrite $L[u](m)$ in the following way:

$$\begin{aligned} & L[u](m) \\ = & \sum_{i \in I_{RD}^m} u^{\nabla_i \Delta_i}(m) \\ & + \sum_{i \in I_{RS}^m} \left(\frac{u^{\Delta_i}(m) - u^{\nabla_i}(m)}{\mu_i(m_i)} + \tilde{B}_i(m)u^{\Delta_i}(m) + \tilde{C}_i(m)u^{\nabla_i}(m)\right) - u^{\nabla_{n+1}}(m). \end{aligned} \tag{13}$$

If $I = I_{RD}^m$, then (13) and (9) imply that

$$L[u](m) = \sum_{i \in I_{RD}^m} u^{\nabla_i \Delta_i}(m) - u^{\nabla_{n+1}}(m) \leq 0,$$

which contradicts (10). Otherwise, we multiply both sides of the equality (13) by $\hat{\mu}(m) > 0$ and use (7) and (9) to obtain that

$$
\begin{aligned}
&\hat{\mu}(m)L[u](m) \\
&= \hat{\mu}(m)\sum_{i\in I^m_{RD}} u^{\nabla_i\Delta_i}(m) \\
&\quad +\hat{\mu}_{-i}(m)\sum_{i\in I^m_{RS}} [(1+\tilde{B}_i(m)\mu_i(m_i))u^{\Delta_i}(m) + (-1+\tilde{C}_i(m)\mu_i(m_i))u^{\nabla_i}(m)] \\
&\quad -\hat{\mu}(m)u^{\nabla_{n+1}}(m) \\
&\leq 0,
\end{aligned}
$$

which contradicts (10) and the proof is done. □

Next we consider the operator which contains the non-derivative term

$$(L+h)[u] := \sum_{i=1}^{n}(u^{\nabla_i\Delta_i} + \tilde{B}_i u^{\Delta_i} + \tilde{C}_i u^{\nabla_i}) - u^{\nabla_{n+1}} + hu.$$

Theorem 4.4 *Let $u \in \mathcal{D}(\Omega)$ satisfy*

$$(L+h)[u] > 0, \qquad in\ \Lambda^{\mathcal{K}}_{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}, \tag{14}$$

and let $\tilde{B}_i$ and $\tilde{C}_i$ satisfy the inequality (12). Moreover, we suppose that

$$h(t) \leq 0, \tag{15}$$

for each $t \in \Lambda^{\mathcal{K}}_{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}$. Then u cannot attain a nonnegative maximum anywhere other than on the parabolic boundary.

Proof. For contradiction, we assume that u attains a nonnegative maximum at a point $m \in \Omega \setminus P\Omega$. By the proof of Theorem 4.3, we know that

$$L[u](m) \leq 0,$$

if u attains its maximum at the point m. Then, together with the condition $h(m)u(m) \leq 0$, we easily see that

$$(L+h)[u](m) = L[u](m) + h(m)u(m) \leq 0,$$

which contradicts (14). $\square$

Corollary 4.5 *If $u \in \mathcal{D}(\Omega)$ satisfies*

$$\sum_{i=1}^{n}(u^{\nabla_i\Delta_i} + \tilde{B}_i u^{\Delta_i} + \tilde{C}_i u^{\nabla_i} + \beta_i u^{\hat{\sigma}_i} + \gamma_i u^{\hat{\rho}_i}) - u^{\nabla_{n+1}} + hu > 0, \tag{16}$$

in $\Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}$. Further, we assume that

$$\begin{cases} 1 + (\tilde{B}_i(t) + \mu_i(t_i)\beta_i(t))\mu_i(t_i) \geq 0, \\ \\ -1 + (\tilde{C}_i(t) - \nu_i(t_i)\gamma_i(t))\mu_i(t_i) \leq 0, \end{cases} \tag{17}$$

for each $t \in \Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}$ which is i-right scattered and $i \in I$, and

$$h + \sum_{i=1}^{n}(\beta_i + \gamma_i) \leq 0, \qquad \text{in } \Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}. \tag{18}$$

Then u cannot attain a nonnegative maximum anywhere other than on the parabolic boundary.

Proof. Using the formula (d) and (g) in the Lemma 2.8, we can obtain the two analogues equalities:

$$\begin{aligned} u(\hat{\sigma}_i(t)) &= u(t) + \mu_i(t_i)u^{\Delta_i}(t), \\ u(\hat{\rho}_i(t)) &= u(t) - \nu_i(t_i)u^{\nabla_i}(t), \end{aligned}$$

for each $t \in \Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}$ and $i \in I$. Substituting these into (16), we obtain

$$\sum_{i=1}^{n}(u^{\nabla_i\Delta_i} + (\tilde{B}_i + \mu_i(t_i)\beta_i)u^{\Delta_i} + (\tilde{C}_i - \nu_i(t_i)\gamma_i)u^{\nabla_i}) - u^{\nabla_{n+1}} + (h + \sum_{i=1}^{n}(\beta_i + \gamma_i))u > 0.$$

Obviously, this operator has the form of (14), and the assumptions (17) and (18) ensure that the inequalities (12) and (15) hold. Consequently, we can use Theorem 4.4 to verify the statement. $\square$

Finally, we establish the weak maximum principles for the parabolic operator and apply it to obtain the uniqueness of solutions for the initial boundary value problem.

Theorem 4.6 *Let $u \in \mathcal{D}(\Omega)$ satisfy*

$$L[u] \geq 0, \qquad in\ \Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}, \tag{19}$$

and we assume that $\tilde{B}_i$ be bounded above and $\tilde{C}_i \leq 0$ satisfy the inequalities (12). Then u attains its maximum on the parabolic boundary, i.e.,

$$\sup_{\Omega} u = \sup_{P\Omega} u. \tag{20}$$

Proof. Since $\tilde{B}_1$ is bounded above, there exists a negative constant α such that

$$\alpha + \tilde{B}_1 < 0, \qquad in\ \Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}. \tag{21}$$

Select any point $\hat{t} \in \mathbb{T}_1$. Then, applying Lemma 2.7 and 2.10, we obtain

$$\begin{aligned} L[\hat{e}_\alpha(t_1,\hat{t})] &= (\hat{e}_\alpha(t_1,\hat{t}))^{\nabla_1\Delta_1} + \tilde{B}_1(\hat{e}_\alpha(t_1,\hat{t}))^{\Delta_1} + \tilde{C}_1(\hat{e}_\alpha(t_1,\hat{t}))^{\nabla_1} \\ &= (\alpha+\tilde{B}_1)\hat{e}_\alpha^{\Delta_1}(t_1,\hat{t}) + \alpha\tilde{C}_1\hat{e}_\alpha(t_1,\hat{t}) \\ &= (\alpha+\tilde{B}_1)\hat{e}_\alpha^{\nabla_1}(\sigma_1(t_1),\hat{t}) + \alpha\tilde{C}_1\hat{e}_\alpha(t_1,\hat{t}) \\ &= (\alpha+\tilde{B}_1)\alpha\hat{e}_\alpha(\sigma_1(t_1),\hat{t}) + \alpha\tilde{C}_1\hat{e}_\alpha(t_1,\sigma_1(t_1))\hat{e}_\alpha(\sigma_1(t_1),\hat{t}) \\ &= \alpha\hat{e}_\alpha(\sigma_1(t_1),\hat{t})[\alpha+\tilde{B}_1+\tilde{C}_1\hat{e}_\alpha(t_1,\sigma_1(t_1))]. \end{aligned} \tag{22}$$

The assumption $\tilde{C}_1 \leq 0$ together with (21), we see that

$$L[\hat{e}_\alpha(t_1,\hat{t})] > 0, \qquad in\ \Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}.$$

Then for each $\varepsilon > 0$, we have

$$L[u+\varepsilon\hat{e}_\alpha(t_1,\hat{t})] = L[u] + \varepsilon L[\hat{e}_\alpha(t_1,\hat{t})] > 0, \tag{23}$$

in $\Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}$, so that

$$\sup_{\Omega}(u+\varepsilon\hat{e}_\alpha(t_1,\hat{t})) = \sup_{P\Omega}(u+\varepsilon\hat{e}_\alpha(t_1,\hat{t})), \tag{24}$$

by applying the Theorem 4.3.

Now we want to show that $\sup\limits_{\Omega} u = \sup\limits_{P\Omega} u$. For contradiction, we assume that $\sup\limits_{\Omega} u > \sup\limits_{P\Omega} u$. Since the time scale $\mathbb{T}_1$ is bounded, this implies that $0 < \hat{e}_\alpha(t_1, \hat{t}) < M$ for some $M > 0$. We set $K = \sup\limits_{\Omega} u - \sup\limits_{P\Omega} u > 0$ and take $\varepsilon = \frac{K}{2M}$, then by applying (24) we can deduce that

$$\begin{aligned}
\sup_{P\Omega}(u + \varepsilon \hat{e}_\alpha(t_1, \hat{t})) &\leq \sup_{P\Omega}(u + \varepsilon M) = \sup_{P\Omega} u + \varepsilon M \\
&= (\sup_{\Omega} u - K) + \frac{K}{2} < \sup_{\Omega} u \\
&\leq \sup_{\Omega}(u + \varepsilon \hat{e}_\alpha(t_1, \hat{t})) = \sup_{P\Omega}(u + \varepsilon \hat{e}_\alpha(t_1, \hat{t})),
\end{aligned}$$

which is a contradiction and the proof is done. $\square$

The above proven maximum principles yields the uniqueness of solutions for the following problem:

$$\begin{cases}
\sum\limits_{i=1}^{n}(u^{\nabla_i \Delta_i} + \tilde{B}_i u^{\Delta_i} + \tilde{C}_i u^{\nabla_i}) - u^{\nabla_{n+1}} = f(t) \quad \textit{on } \Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}, \\
u(t) = g(t) \quad \textit{on } B, \\
u(t) = h(t) \quad \textit{on } S.
\end{cases} \tag{25}$$

Theorem 4.7 *Suppose that the assumptions of Theorem 4.6 holds. If u_1 and u_2 are solutions of the initial boundary value problem (25), then $u_1 \equiv u_2$.*

Proof. First of all, we define the auxiliary function $v = u_1 - u_2$. Since both u_1 and u_2 are solutions of (25), this implies that

$$\begin{cases}
\sum\limits_{i=1}^{n}(v^{\nabla_i \Delta_i} + \tilde{B}_i v^{\Delta_i} + \tilde{C}_i v^{\nabla_i}) - v^{\nabla_{n+1}} = 0 \quad \textit{on } \Lambda_{\mathcal{K}}^{\mathcal{K}} \times ([0,T]_{\mathbb{T}_{n+1}})_{\mathcal{K}}, \\
v(t) = 0 \quad \textit{on } P\Omega
\end{cases} \tag{26}$$

Obviously, we know that $-v$ is also a solution of (26). Then by applying Theorem 4.6, we have that

$$\sup_{\Omega} v = \sup_{P\Omega} v = 0 \qquad and \qquad \sup_{\Omega}(-v) = \sup_{P\Omega}(-v) = 0.$$

It follows that

$$v(t) \leq 0 \qquad and \qquad -v(t) \leq 0,$$

for each $t \in \Omega$. Consequently, we get the conclusion that $v = u_1 - u_2 = 0$. $\square$

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國科會補助計畫衍生研發成果推廣資料表

日期:2011/10/14

<table>
<tr><td rowspan="3">國科會補助計畫</td><td>計畫名稱: 在time scales上的衝擊動態方程的週期邊界值問題</td></tr>
<tr><td>計畫主持人: 符聖珍</td></tr>
<tr><td>計畫編號: 99-2115-M-004-002- 學門領域: 微分方程</td></tr>
<tr><td colspan="2">無研發成果推廣資料</td></tr>
</table>

99 年度專題研究計畫研究成果彙整表

計畫主持人：符聖珍	計畫編號：99-2115-M-004-002-
計畫名稱：在 time scales 上的衝擊動態方程的週期邊界值問題	

成果項目			量化			單位	備註（質化說明：如數個計畫共同成果、成果列為該期刊之封面故事...等）
			實際已達成數（被接受或已發表）	預期總達成數(含實際已達成數)	本計畫實際貢獻百分比		
國內	論文著作	期刊論文	0	1	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%		
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（本國籍）	碩士生	1	1	100%	人次	
		博士生	0	0	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		
國外	論文著作	期刊論文	0	0	100%	篇	
		研究報告/技術報告	0	0	100%		
		研討會論文	0	0	100%		
		專書	0	0	100%	章/本	
	專利	申請中件數	0	0	100%	件	
		已獲得件數	0	0	100%		
	技術移轉	件數	0	0	100%	件	
		權利金	0	0	100%	千元	
	參與計畫人力（外國籍）	碩士生	0	0	100%	人次	
		博士生	0	0	100%		
		博士後研究員	0	0	100%		
		專任助理	0	0	100%		

其他成果 (無法以量化表達之成果如辦理學術活動、獲得獎項、重要國際合作、研究成果國際影響力及其他協助產業技術發展之具體效益事項等，請以文字敘述填列。)	無

	成果項目	量化	名稱或內容性質簡述
科教處計畫加填項目	測驗工具(含質性與量性)	0	
	課程/模組	0	
	電腦及網路系統或工具	0	
	教材	0	
	舉辦之活動/競賽	0	
	研討會/工作坊	0	
	電子報、網站	0	
	計畫成果推廣之參與（閱聽）人數	0	

國科會補助專題研究計畫成果報告自評表

請就研究內容與原計畫相符程度、達成預期目標情況、研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）、是否適合在學術期刊發表或申請專利、主要發現或其他有關價值等，作一綜合評估。

1. 請就研究內容與原計畫相符程度、達成預期目標情況作一綜合評估

■達成目標

□未達成目標（請說明，以 100 字為限）

　　□實驗失敗

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　　□其他原因

說明：

2. 研究成果在學術期刊發表或申請專利等情形：

論文：□已發表 ■未發表之文稿 □撰寫中 □無

專利：□已獲得 □申請中 ■無

技轉：□已技轉 □洽談中 ■無

其他：（以 100 字為限）

3. 請依學術成就、技術創新、社會影響等方面，評估研究成果之學術或應用價值（簡要敘述成果所代表之意義、價值、影響或進一步發展之可能性）（以 500 字為限）

In recent years, the study of dynamic equations on time scales has received a lot of attentions since it not only can unify the calculation of difference and differential equations but also has various applications. In this project, we study the maximum principles for the elliptic and parabolic dynamic equations on multi-dimensional time scales.