國立政治大學應用數學系

碩士學位論文

On K Stability and Tropical Geometry

K 穩定性與熱帶幾何之研究

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謝辭

隨著口試的結束，政大六年的生活也來到了尾聲，雖然即將畢業是令人喜悅的，但想到馬上要離開政大，卻有著更多的不捨。回想起這六年間的種種往事，心中充滿了太多的感激。

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Abstract

In this thesis, we analyze K stability on compact Fano hypersurfaces from K energy. We first represent the K energy into an explicitly formula. Then we compute the derivative by using some analytic techniques. Furthermore, we introduce some structures of tropical geometry to analyze the main result. Finally, we give some examples of compact Fano hypersurface to test and verify the formula we get.

中文摘要

在這篇論文中，我們從K energy的角度探討緊緻法諾超平面上的K穩定性。首先，我們給K energy一個較明確的型式，接著再透過分析的手法求解其導函數。後續，我們引進熱帶幾何的結構來重新分析主要的結果，最後給一些法諾超平面的實例，驗證我們所得到的公式。

Contents

1 Introduction

Definition 1.1 *Let M be a Hermitian complex manifold with Hermitian metric g. In local coordinates* $(z_1, \cdots, z_n)$, *g can be written in th form*

$$g = \sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \otimes d\bar{z}_j$$

where $\{g_{i\bar{j}}\}$ *is a positive definite Hermitian matrix function. The associated Kähler form defined by*

$$\omega = \frac{i}{2} \sum_{i,j=1}^{n} g_{i\bar{j}} dz_i \wedge d\bar{z}_j,$$

which is closed, i.e. $d\omega = 0$. *A complex manifold M equipped with a Kähler metric is called a Kähler manifold.*

Definition 1.2 *The Kähler metric is called a Kähler-Einstein metric if its Ricci curvature form is a constant multiple of its Kähler form.*

In 1954, E. Calabi conjectured that a compact Kähler manifold M has a unique Kähler metric in the same class whose Ricci form is any given 2-form representing the first Chern class $c_1(M)$. In particular, the conjecture closely related to the existence of Kähler-Einstein metrics on a compact Kähler manifold M with its first Chern class $c_1(M)$ definite.

The question was proved for negative first Chern classes independently by Thierry Aubin and Shing-Tung Yau in 1976 (cf, [1], [18]). When the first Chern class is zero, it was proved by Yau in 1977 as an easy consequence of the Calabi conjecture [18]. Therefore, Kähler-Einstein metrics exist on the underlying manifold as the first chern class $c_1(M)$ being zero or negative.

The uniqueness of this two cases was proved by Calabi himself. In 1986, Bando and Mabuchi proved the uniqueness of Kähler-Einstein metrics on compact Fano manifolds. A Fano manifold is a Kähler manifolds with positive first Chern class.

So the remaining case is the existence of Kähler-Einstein metrics of constant scalar curvature.

In 1957, Matsushima proved that a necessary condition for the existence of a Kähler–Einstein metric is that the Lie algebra (M) of holomorphic fields is reductive [13]. Yau conjectured that when the first Chern class $c_1(M)$ is positive, a Kähler variety has a Kähler–Einstein metric if and only if it is stable in the sense of geometric invariant theory. In 1983, Futaki[6] proved that the Futaki invarient f_M is zero if M has a Kähler-Einstein metrics. The Futaki invarient f_M is a character of the Lie algebra $\eta(M)$. In 1988, D. Burns and P. De Bartolomeis proved that the projective bundles does not admit a Kähler metric with constant scalar curvature(cf, [2], [8], [15]). In 1989, Gang Tian proved that any complex surface M with $c_1(M) > 0$ has a Kähler–Einstein metric if and only if $\eta(M)$ is reductive.

In Tian[17] and Donaldson[5], the notion of K stability was introduced. In Mabuchi[12], the definition of K stability is related to K energy.

Definition 1.3 *Let M be a compact Kähler manifolds with positive first Chern class $c_1(M)$. Let ω_0 and ω_1 be any two Kähler metrics in $c_1(M)$, there is a smooth function φ, unique up to the addition of constants, satisfying***:**

$$\omega_1 = \omega_0 + \frac{i}{2\pi}\partial\overline{\partial}\varphi.$$

Put $\omega_s = \omega_0 + s\frac{i}{2\pi}\partial\overline{\partial}\varphi$ and defined

$$\mathcal{M}(\omega_0, \omega_1) = -\frac{1}{V}\int_0^1 (\int_M \varphi(R(\omega_s) - n)\omega_s^n)ds,$$

where $R(\omega_s)$ is the scalar curvature of the metric, n is the complex dimension of M, and V is the volume of M with respect to ω_0. The functional $\mathcal{M}$ is called the K energy.

Proposition 1.4 *Using the notation as above, we have*:

(a) $\mathcal{M}(\omega_0, \omega_1) = -\mathcal{M}(\omega_1, \omega_0)$,

(b) $\mathcal{M}(\omega_0, \omega_1) + \mathcal{M}(\omega_1, \omega_2) = \mathcal{M}(\omega_0, \omega_2)$,

where ω_0, ω_1, ω_2 are the Kähler metrics in $c_1(M)$.

***Proof*.**

(a) By the definition of K energy, we have $\omega_1 = \omega_0 + \frac{i}{2\pi}\partial\overline{\partial}\varphi$ and $\omega_s = \omega_0 + s\frac{i}{2\pi}\partial\overline{\partial}\varphi$. Set $s = 1-t$, we can get $\omega_0 = \omega_1 - \frac{i}{2\pi}\partial\overline{\partial}\varphi = \omega_1 + \frac{i}{2\pi}\partial\overline{\partial}(-\varphi)$ and $\omega_{1-t} = \omega_0 + (1-t)\frac{i}{2\pi}\partial\overline{\partial}\varphi = \omega_1 + t\frac{i}{2\pi}\partial\overline{\partial}(-\varphi)$

$$\begin{aligned}
\mathcal{M}(\omega_0, \omega_1) &= -\frac{1}{V}\int_0^1 \Big(\int_M \varphi(R(\omega_s) - n)\omega_s^n\Big) ds \\
&= -\frac{1}{V}\int_1^0 \Big(\int_M \varphi(R(\omega_{1-t}) - n)\omega_{1-t}^n\Big) - dt \\
&= -\frac{1}{V}\int_0^1 \Big(\int_M \varphi(R(\omega_{1-t}) - n)\omega_{1-t}^n\Big) dt \\
&= \frac{1}{V}\int_0^1 \Big(\int_M (-\varphi)(R(\omega_{1-t}) - n)\omega_{1-t}^n\Big) dt \\
&= -\mathcal{M}(\omega_1, \omega_0).
\end{aligned}$$

(b) Let $\omega_1 = \omega_0 + \frac{i}{2\pi}\partial\overline{\partial}\varphi_1$, $\omega_s = \omega_0 + s\frac{i}{2\pi}\partial\overline{\partial}\varphi_1$ and $\omega_2 = \omega_1 + \frac{i}{2\pi}\partial\overline{\partial}\varphi_2$, $\omega_{t+1} = \omega_1 + t\frac{i}{2\pi}\partial\overline{\partial}\varphi_2$. Then set $u = \frac{s}{2}$, $v = \frac{t+1}{2}$, we have $\omega_2 = \omega_1 + \frac{i}{2\pi}\partial\overline{\partial}\varphi_2 = \omega_0 + \frac{i}{2\pi}\partial\overline{\partial}\varphi_1 + \frac{i}{2\pi}\partial\overline{\partial}\varphi_2 = \omega_1 + \frac{i}{2\pi}\partial\overline{\partial}(\varphi_1 + \varphi_2)$ and $\omega_{2u} = \omega_0 + 2u\frac{i}{2\pi}\partial\overline{\partial}\varphi_1$, $\omega_{2v} = \omega_1 + (2v-1)\frac{i}{2\pi}\partial\overline{\partial}\varphi_2 = \omega_0 + \frac{i}{2\pi}\partial\overline{\partial}\varphi_1 + (2v-1)\frac{i}{2\pi}\partial\overline{\partial}\varphi_2$.

$$
\begin{aligned}
& \mathcal{M}(\omega_0,\omega_1)+\mathcal{M}(\omega_1,\omega_2) \\
= \;& -\frac{1}{V}\int_0^1 \Big(\int_M \varphi_1(R(\omega_s)-n)\omega_s^n\Big)ds - \frac{1}{V}\int_0^1 \Big(\int_M \varphi_2(R(\omega_{t+1})-n)\omega_{t+1}^n\Big)dt \\
= \;& -\frac{1}{V}\int_0^{\frac{1}{2}} \Big(\int_M 2\varphi_1(R(\omega_{2u})-n)\omega_{2u}^n\Big)du - \frac{1}{V}\int_{\frac{1}{2}}^1 \Big(\int_M 2\varphi_2(R(\omega_{2v})-n)\omega_{2v}^n\Big)dv \\
= \;& -\frac{1}{V}\int_0^1 \Big(\int_M (\varphi_1+\varphi_2)(R(\omega_{2s})-n)\omega_{2s}^n\Big)ds \\
= \;& \mathcal{M}(\omega_0,\omega_2).
\end{aligned}
$$

❑

In this thesis, we setup notations: Let ω be the Kähler form of the Fubini-Study metric on $\mathbb{CP}^n$. Let M be a hypersurface in $\mathbb{CP}^n$ defined by the polynomial F = 0 of degree d. To make sure that M is a Fano manifold, d must less or equal to n. Let $\lambda_0,\cdots,\lambda_n$ be integers such that $\sum_{i=0}^{n}\lambda_i = 0$. Let F_t be the polynomial defined by

$$F_t(Z_0,\cdots,Z_n) = F(t^{-\lambda_0}Z_0,\cdots,t^{-\lambda_n}Z_n),$$

and let M_t be the hypersurface defined by the zero set of F_t. Let $\sigma(t)$ be a one parameter family of automorphisms of $\mathbb{CP}^n$ which can be written as

$$\sigma(t)[Z_0,\cdots,Z_n] = [t^{\lambda_0}Z_0,\cdots,t^{\lambda_n}Z_n].$$

Consider that $\sigma(t)$ is generated by the holomorphic vector field $X = \sum_{i=0}^{n}\lambda_i Z_i \frac{\partial}{\partial Z_i}$, M_t is the image in geometry sence. The degeneration of M by X is defined as the hypersurface in $\mathbb{C}\times\mathbb{CP}^n$ by $G(t,Z) = F_t(Z) = 0$. The central fiber of the degeneration is defined as the intersection of the degeneration with the set $\{0\}\times\mathbb{CP}^n$, excluding the factor $t = 0$. Using these automorphisms, we can define a family of Kähler forms $\omega_t = \sigma(t)^*\omega$ on M such that $\alpha\omega_t \in c_1(M)$, where α is a rational number. Tian[17] showed that $\lim_{t\to 0} t\frac{d}{dt}\mathcal{M}(\omega,\omega_t) = A$ exists, where $\mathcal{M}(\omega,\omega_t)$ be the K energy with respect to the metric $\alpha\omega$ and $\alpha\omega_t$. Clearly, both $(n-d+1)\omega$

and $(n-d+1)\omega_t$ are Kähler forms of M in $c_1(M)$. Define $\mathcal{M}(t) = \mathcal{M}((n-d+1)\omega, (n-d+1)\omega_t)$. Mabuchi[12] showed that $\mathcal{M}(t)$ has a lower bound if M admit a Kähler–Einstein metric.

Proposition 1.5 *(Tian) Using the notation as above, we have*:

$$t\frac{d}{dt}\mathcal{M}(t) = \frac{2(n-1)}{d}\int_{M_t}(Ric(\omega|_{M_t}) - (n-d+1)\omega|_{M_t})\theta\omega^{n-2},$$

where θ is defined as

$$\theta = -\frac{\sum_{i=0}^{n}\lambda_i|Z_i|^2}{\sum_{i=0}^{n}|Z_i|^2},$$

and $Ric(\omega|_{M_t})$ is the Ricci form of $\omega|_{M_t}$.

Definition 1.6 *We say that M is K stable if for any holomorphic vector field X on $\mathbb{CP}^n$ with $\lambda_0, \cdots, \lambda_n$ integers and $\lambda_0^2 + \cdots + \lambda_n^2 \neq 0$,*

$$\lim_{t\to 0} t\frac{d}{dt}\mathcal{M}(\omega, \omega_t) < 0.$$

If the above quantity is nonpositive for all vectors X on $\mathbb{CP}^n$, we say M is K semistable.

In 1992, Ding and Tian[3] proved that a cubic surface in $\mathbb{CP}^3$ has a Kähler-Einstein orbifold metric if it is semistable in the sence of Mumford. Tian[17] showed that a Kähler-Einstein metric exists on a compact Kähler manifold M with positive first Chern class $c_1(M)$ and without any nontrivial holomorphic field if and only if the K energy is proper. In particular, if M has no nonzero holomorphic vector field, M is K stable.

Donaldson[5] gives a very similar definition of K stability in algebraic geometry sence.

Definition 1.7 *The pair $(M, \mathcal{L})$ is K stable if for each test configuration for $(M, \mathcal{L})$ the Futaki invariant of the induced action on $(M_0, \mathcal{L}|_{M_0})$ is less than or equal to zero, with equality if and only if the configuration is a product configuration.*

Donaldson showed that if $(M, \mathcal{L})$ is a toric variety such that the Mabuchi functional is bounded below on the invariant metrics and any minimising sequence has a K convergent subsequence, then $(M, \mathcal{L})$ is K stable with respect to toric degenerations. In 2005, Donaldson proved that the Kähler metric with constant scalar curvature implies K semistability. In the same year, he proved that the Kähler metric with constant scalar curvature minimizes the Mabuchi function.

In order to state the main result, we make a little change for some notations: let M be defined by the zeros of the polynomial

$$F(Z_0, \cdots, Z_n) = \sum_{i=0}^{p} a_i Z_0^{\alpha_0^{(i)}} \cdots Z_n^{\alpha_n^{(i)}} \tag{1.1}$$

of degree d. Let $(\lambda_0, \cdots, \lambda_n)$ be rational numbers satisfying $\sum_{i=0}^{n} \lambda_i = 0$. Let

$$\lambda = \max_{0 \le i \le p} (\sum_{k=0}^{n} \lambda_k \alpha_k^{(i)}). \tag{1.2}$$

Let

$$\psi(x_0, \cdots, x_n) = \min_{0 \le i \le p} (-\sum_{k=0}^{n} \lambda_k \alpha_k^{(i)} + \sum_{k=0}^{n} \alpha_k^{(i)} x_k), \tag{1.3}$$

and let

$$\psi_i(x) = \psi(0, \cdots, \underset{\substack{\uparrow \\ i\text{-}th}}{x}, \cdots, 0) \tag{1.4}$$

Remark. In this thesis, K stable means either K stable and K semistable. On the other hand, for the application in Geometric Invariant Theory, we just need to assume that t is a real number and $\lambda_0, \cdots, \lambda_n$ are rational numbers.

Theorem 1.8 *For generic* $(\lambda_0, \cdots, \lambda_n)$*, we have*

$$\begin{aligned} & \lim_{t\to 0} t\frac{d}{dt}\mathcal{M}(t) \\ = \quad & \frac{2}{d}(-\frac{\lambda(d-1)(n+1)}{n} + \sum_{i=0}^{n}\int_0^\infty \psi_i'(x)(\psi_i'(x)-1)dx). \end{aligned} \tag{1.5}$$

The purpose in this thesis is to find an effective way to verify the K stability for hypersurface. Since the K energy is the nonlinear version of the Futaki invariant, it is harder than find an effective way to compute the Futaki invariant. In [3] or [17], if the central fiber is normal, the quantity A is the real part of the corresponding Futaki invariant. The limit in theorem 1.8 depends not only on the central fiber, but also on the whole degeneration F_t. We represent the K energy into an explicitly formula in section 3. Then we compute the limit of $t\frac{d}{dt}\mathcal{M}(t)$ by using some analytic manners and a result of Phong and Sturm[14] in section 4.

Note that $(1.1), (1.2), (1.3)$ and (1.4) be considered in the tropical semiring. We will introduce some structures of tropical geometry in section 2.

2 Tropical Geometry

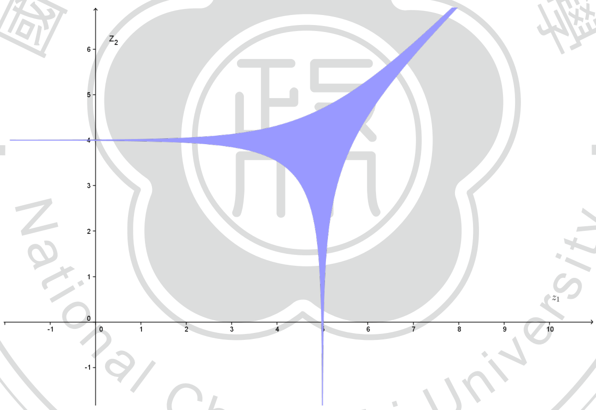
In this section, we will introduce some structures of tropical geometry.

For a complex plane curve C, we restrict it to the open subset $(\mathbb{C}^*)^2$ of the (affine or projective) plane and then map it to the real plane by the map

$$\begin{aligned} \mathrm{Log} : (\mathbb{C}^*)^2 &\longrightarrow \mathbb{R}^2 \\ z = (z_1, z_2) &\longmapsto (x_1, x_2) := (\log|z_1|, \log|z_2|). \end{aligned}$$

The image $A = \mathrm{Log}(C \cap (\mathbb{C}^*)^2)$ is called the amoeba of the given curve C.

***Example*.**



$C = \{z \in (\mathbb{C}^*)^2 | \exp^{-5} z_1 + \exp^{-4} z_2 = 1\}$

Figure 1

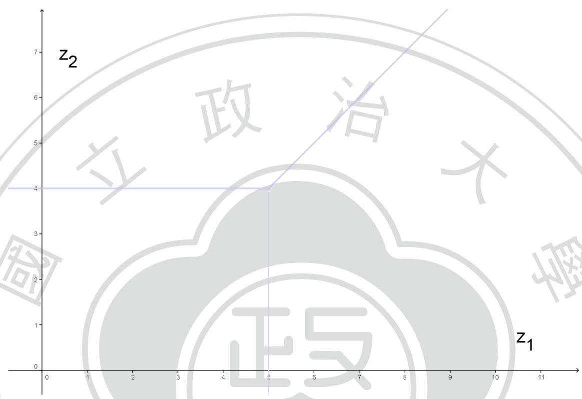
❏

In the example above, the curve C contains exactly one point whose z_1-coordinate is zero, namely $(0, e^4)$. Since $\log 0 \to -\infty$ as t tends to 0, a small neighborhood of the point $(0, e^4)$ is mapped by Log to the tentacle of the amoeba A pointing to the left. Similarly, a small neighborhood of $(e^5, 0)$ mapped by Log to the tentacle pointing down, and point of the form $(z, e - e^5 z)$ with $|z| \to \infty$ to the tentacle pointing to the upper left.

Consider the maps

$$\begin{aligned} \mathrm{Log}_t \,:\, (\mathbb{C}^*)^2 &\longrightarrow \mathbb{R}^2 \\ (z_1, z_2) &\longmapsto (-\log_t |z_1|, -\log_t |z_2|) = (-\frac{\log |z_1|}{\log t}, -\frac{\log |z_2|}{\log t}) \end{aligned}$$

for small $t \in \mathbb{R}$. Then the image $\Gamma = \mathrm{Log}_t(C \cap (\mathbb{C}^*)^2)$ is similar to amoeba of C, but the width of A will shrink to zero as t tends to zero. We called Γ the tropical curve determined by C.



$C = \{z \in (\mathbb{C}^*)^2 \big| \, e^{-5}z_1 + e^{-4}z_2 = 1\}$

Figure 2. The tropical curve corresponding to the amoeba in figure 1.

The curve with graph showed in Figure 2 is not unique. So we consider not only the curve $C = \{z \in (\mathbb{C}^*)^2 \big| \, e^{-5}z_1 + e^{-4}z_2 = 1\}$ but the family of curves $C_t = \{z \in \mathbb{C}^2 \big| \, t^5 z_1 + t^4 z_2 = 1\}$ for small $t \in \mathbb{R}$. This family has the property that C_t passes through $(0, t^{-4})$ and $(t^{-5}, 0)$ for all t, and hence all $\log_t(C_t \cap (\mathbb{C}^*)^2)$ have their horizontal and vertical tentacles at $z_2 = 4$ and $z_1 = 5$, respectively. So if we take the limit as t tends to 0, we shrink the width of amoeba to zero. We called this the tropical curve determined by the family C_t.

Definition 2.1 *A formal series of the form* $\sum_{q \in \mathbb{Q}} a_q t^q, a_q \in \mathbb{C}$ *satisfying:*

(i) the set $\{q \in \mathbb{Q} \big| \, a_q \neq 0\}$ *is bounded below,*

(ii) the denominators of q is a finite set

is called a Puiseux series or a fractional power series. A field K of Puiseux series is a collection of Puiseux series.

Given $a \in K$ with the expression $a = \sum_{q\in\mathbb{Q}} a_q t^q$, denote the valuation of a by

$$\operatorname{val} a = \inf\{q \in \mathbb{Q} \big| \, a_q \neq 0\} = \min\{q \in \mathbb{Q} \big| \, a_q \neq 0\}.$$

For any element $a = \sum_{q\in\mathbb{Q}} a_q t^q \in K$, as t small enough, a approximate to the term with the smallest exponent, i.e. $a_{\operatorname{val} a} t^{\operatorname{val} a}$. So applying the map $\log_t$ we get

$$\log_t |a| \approx \log_t |a_{\operatorname{val} a} t^{\operatorname{val} a}| = \operatorname{val} a + \log_t |a_{\operatorname{val} a}| \approx \operatorname{val} a$$

for small t. Using this approximate, the map Log_t and take the limit for $t \to 0$ is correspond to the map

$$\begin{aligned} \operatorname{Val} : (K^*)^2 &\longrightarrow \mathbb{R}^2 \\ (z_1, z_2) &\longmapsto (x_1, x_2) := (-\operatorname{val} z_1, -\operatorname{val} z_2). \end{aligned}$$

Hence, we can now give a severe definition of plane tropical curves :

Definition 2.2 *A plane tropical curve is a subset of* $\mathbb{R}^2$ *of the form Val*$(C \cap (K^*)^2)$*, where* C *is a plane algebraic curve in* K^2*. (Strictly speaking we should take the closure of Val*$(C \cap (K^*)^2)$ *in* $\mathbb{R}^2$ *since the image of the valuation map Val is by definition contained in* $\mathbb{Q}^2$*.)*

Note that this definition is now purely algebraic and is not concerned with any limit taking processes.
For example, consider the curve $C = \{z \in K^2 \big| \, t^5 z_1 + t^4 z_2 = 1\}$. If $(z_1, z_2) \in C \cap (K^*)^2$ then $\operatorname{Val}(z_1, z_2)$ can give three different kinds of result :

1. If $\operatorname{val} z_1 > -5$ then the valuation of $z_2 = t^{-4} - tz_1$ is -4 since all exponents of t in tz_1 are bigger than -4. Hence these points map precisely to the left edge of the tropical curve.

2. If $\operatorname{val} z_2 > -4$ then the valuation of $z_1 = t^{-5} - t^{-1}z_2$ is -5 since all exponents of t in $t^{-1}z_2$ are bigger than -5. Hence these points map precisely to the down edge of the tropical curve.

3. If $\operatorname{val} z_1 \leq -5$ and $\operatorname{val} z_2 \leq -4$ then the equation $t^5 z_1 + t^4 z_2 = 1$ shows that $\operatorname{val}(t^5 z_1) = \operatorname{val}(t^4 z_2)$, i.e. $\operatorname{val} z_1 = \operatorname{val} z_2 + 1$. This leads to the upper right edge of the tropical curve.

So we can get the same graph by this definition.

Let $C \subset K^2$ be a plane algebraic curve given by the polynimial equation

$$C = \{(z_1, z_2) \in K^2 \big| \, f(z_1, z_2) := \sum_{i,j \in \mathbb{N}} a_{ij} z_1^i z_2^j = 0\}$$

for some $a_{ij} \in K$ of which only finite many are nonzero. Note that the valuation of a summand of $f(z_1, z_2)$ is

$$\operatorname{val}(a_{ij} z_1^i z_2^j) = \operatorname{val} a_{ij} + i \operatorname{val} z_1 + j \operatorname{val} z_2.$$

Now if (z_1, z_2) is a point of C then all these summands add up to zero. In particular, the lowest valuation of these summands must occur at least twice since otherwise the corresponding terms in the sum could not cancel. For the corresponding point $(x_1, x_2) = \operatorname{Val}(z_1, z_2) = (-\operatorname{val} z_1, -\operatorname{val} z_2)$ of the tropical curve, this means that in the expression

$$g(x_1, x_2) := \max\{ix_1 + jx_2 - \operatorname{val} a_{ij} \big| \, (i, j) \in \mathbb{N}^2 \text{ with } a_{ij} \neq 0\} \tag{2.1}$$

the maximum is taken on at least twice. It follows that the tropical curve determined by C is contained in the "corner locus" of this convex piecewise linear function g, the corner locus is the locus where g is not differentiable.

Theorem 2.3 *(Kapranov) The closure of the amoeba $A \subset \mathbb{R}^2$ coincides with the corner locus of the convex piecewise linear function g. If the valuation val : $K^* \to \mathbb{R}$ is surjective, then A coincides with the corner locus of g.*

Remark. Kapranov's theorem shows that the tropical curve determined by C is precisely the corner locus of g.

For example, let us consider the curve $C = \{(z_1, z_2) \in K^2 \big| \, t^5 z_1 + t^4 z_2 = 1\} \subset K^2$ again. The corresponding convex piecewise linear function with respect to C is $g(x_1, x_2) = \max\{x_1 - 5, x_2 - 4, 0\}$. Figure 3 shows that the relation between tropical curve and the convex piecewise linear function g.

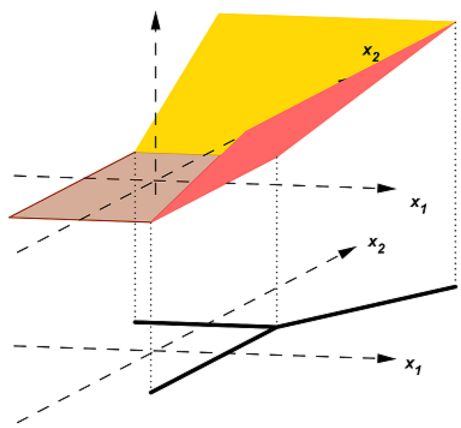


Figure 3. A tropical curve as the corner locus of a convex piecewise linear function.

In order to represent these piecewise linear functions as the notation of the original polynomial, we need to introduce two operators.

Definition 2.4 *Let $\mathbb{T} := \mathbb{R} \cup \{-\infty\}$, we define operators $\oplus : \mathbb{T} \times \mathbb{T} \to \mathbb{T}$, and $\odot : \mathbb{T} \times \mathbb{T} \to \mathbb{T}$ by*

$$\begin{aligned} x \oplus y &:= \max\{x, y\}, \\ x \odot y &:= x + y. \end{aligned}$$

The operator $\oplus$ is called the tropical addition, and the operator $\odot$ is called the tropical multiplication.

Since for each $a \in \mathbb{T}$, $a \oplus (-\infty) = a$, $x \odot 0 = x$, so $\mathbb{T}$ has the additive identity element $-\infty$ and the multiplicative identity element 0.

Definition 2.5 *For each $a \in \mathbb{T}$, $n \in \mathbb{Z}$, define*

$$a^{\odot n} := n \times a.$$

And define tropical division to be their usual subtraction:

$$x \oslash y := x - y.$$

Moreover, define

$$\bigoplus_{i=1}^{n} a_i := \max\{a_1, \cdots, a_n\},$$
$$\bigodot_{i=1}^{n} a_i := a_1 + a_2 + \cdots + a_n.$$

Definition 2.6 *A semiring is a set S equipped with two binary operations "+" and "·", called addition and multiplication, respectively, such that:*

(i) $(S, +)$ *is a commutative monoid with identity element 0.*

(ii) $(S, \cdot)$ *is a monoid with identity element 1.*

(iii) The multiplication is distributive with respect to the addition.

(iv) Multiplication by 0 annihilates S, i.e. for all $a \in S$, $a \cdot 0 = 0 \cdot a = 0$.

Remark. $(\mathbb{T}, \oplus, \odot) = (\mathbb{R} \cup \{-\infty\}, \max, +)$ is a semiring.

Using the notation above, (2.1) can be written as

$$g(x_1, x_2) = \bigoplus_{i,j} (-\mathrm{val}\, a_{ij}) \odot x_1^{\odot i} \odot x_2^{\odot j}.$$

We call this expression the tropicalization of the original polynomial f. It can be considered as a polynomial in the tropical semiring. For example, the tropicalization of the polynomial $t^5 z_1 + t^4 z_2 = 1$ is just

$$(-5) \odot x_1 \oplus (-4) \odot x_2 \oplus 0 = \max\{x_1 - 5, x_2 - 4, 0\}.$$

Now, we generalize this concept into the polynomial with n variables: $f(z) = f(z_1, \cdots, z_n) = \sum_{i=1}^{p} a_i z_1^{i_1} \cdots z_n^{i_n}$. The tropicalization of f is

$$\begin{aligned} g(x) = g(x_1, \cdots, x_n) \quad &= \max_{1 \le i \le p} \{i_1 x_1 + \cdots + i_n x_n - \mathrm{val}\, a_i\} \\ &= \bigoplus_{i=1}^{p} (-\mathrm{val}\, a_i) \odot x_1^{\odot i_1} \odot \cdots \odot x_n^{\odot i_n} \end{aligned}$$

where $x_j = -\mathrm{val}\, z_j$, $i_j \in \mathbb{N}$, for all $1 \le j \le n$, $1 \le i \le p$.

In section 1, we setup the notations: M be defined by the zeros of the polynomial

$$F(Z_0, \cdots, Z_n) = \sum_{i=0}^{p} a_i Z_0^{\alpha_0^{(i)}} \cdots Z_n^{\alpha_n^{(i)}}$$

of degree d. Let $(\lambda_0, \cdots, \lambda_n)$ be rational numbers satisfying $\sum_{i=0}^{n} \lambda_i = 0$. Let

$$\lambda = \max_{0 \le i \le p} (\sum_{k=0}^{n} \lambda_k \alpha_k^{(i)}).$$

Let

$$\psi(x_0, \cdots, x_n) = \min_{0 \le i \le p} (-\sum_{k=0}^{n} \lambda_k \alpha_k^{(i)} + \sum_{k=0}^{n} \alpha_k^{(i)} x_k),$$

and let

$$\psi_k(x) = \psi(0, \cdots, \underset{\substack{\uparrow \\ k\text{-}th}}{x}, \cdots, 0)$$

These notations can be considered as the equation in the tropical semiring. Let M be defined by the zeros of the polynomial

$$F = \bigoplus_{i=0}^{p} (-\operatorname{val} a_i) \odot y_0^{\odot \alpha_0^{(i)}} \odot \cdots \odot y_n^{\odot \alpha_n^{(i)}}$$

of degree d, where $y_j = -\operatorname{val} Z_j$, for all $j = 0, \cdots, n$. Let $(\lambda_0, \cdots, \lambda_n)$ be rational numbers satisfying $\bigodot_{i=0}^{n} \lambda_i = 0$. Let

$$\lambda = \bigoplus_{i=0}^{p} \lambda_0^{\odot \alpha_0^{(i)}} \odot \cdots \odot \lambda_n^{\odot \alpha_n^{(i)}}.$$

Let

$$\psi(x_0, \cdots, x_n) = -\bigoplus_{i=0}^{p} \left((\lambda_0^{\odot \alpha_0^{(i)}} \odot \cdots \odot \lambda_n^{\odot \alpha_n^{(i)}}) \oslash (x_0^{\odot \alpha_0^{(i)}} \odot \cdots \odot x_n^{\odot \alpha_n^{(i)}}) \right),$$

and let

$$\psi_k(x) = -\bigoplus_{i=0}^{p} \left((\lambda_0^{\odot \alpha_0^{(i)}} \odot \cdots \odot \lambda_n^{\odot \alpha_n^{(i)}}) \oslash x^{\odot \alpha_k^{(i)}} \right).$$

3 An explicit formula for the K energy

In this section, we analyze the K energy of smooth hypersurfaces of $\mathbb{CP}^n$ and get an explicit formula.

For the sake of completeness, we need a lemma for Tian[16] stated.

Lemma 3.1 *Let M be the smooth hypersurface defined as the zero of $\{F = 0\}$. We use ω to denote the Fubini-Study metric on $\mathbb{CP}^n$ as well as the Kähler form on M, which is the restriction of ω on M. Let*

$$\varphi = \log \frac{|\nabla F|^2}{(\sum_{i=0}^n |Z_i|^2)^{(d-1)}}, \tag{3.1}$$

where $[Z_0, \cdots, Z_n]$ is the homogeneous coordinate in $\mathbb{CP}^n$. Then we have

$$Ric(\omega) - (n - d + 1)\omega = -\frac{i}{2\pi}\partial\bar{\partial}\varphi. \tag{3.2}$$

Proof. For $i = 0, 1, \cdots, n$, set

$$U_i = \{[Z_0, \cdots, Z_n] \big| |Z_i| > \frac{1}{2}|Z_j|, j = 0, 1, \cdots, j \neq i\}$$

be open set in $\mathbb{CP}^n$. $\cup_{i=0}^n U_i = \mathbb{CP}^n$. We just prove this lemma on U_0. Let $z = (z_1, \cdots, z_n)$ where $z_i = \dfrac{Z_i}{Z_0}$ for $i = 1, \cdots, n$, which is the local coordinate system of U_0. Since (M, g, J) be a complex manifold with Hermitian metric g, we have $\omega(u, v) = g(Ju, v)$. Using the coordinate system, the Fubini-Study metric can be written as

$$\omega = \frac{i}{2\pi}\sum_{j,k=1}^n g_{j\bar{k}} dz_j \wedge d\bar{z}_k = \frac{i}{2\pi}\sum_{j,k=1}^n (\frac{\delta_{jk}}{1 + |z|^2} - \frac{z_k \bar{z}_j}{(1 + |z|^2)^2}) dz_j \wedge d\bar{z}_k, \tag{3.3}$$

where $|z|^2 = \sum_{i=1}^n |z_i|^2$. On any open set V in U_0, since the equation $F = 0$, we can solve z_1. Write

$$z_1 = z_1(z_2, \cdots, z_n) \tag{3.4}$$

for a holomorphic function z_1. Under the local coordinate system $(z_2, \cdots, z_n)$, the Kähler form ω on V can be written as

$$\omega = \frac{i}{2\pi}\sum_{j,k=2}^{n}\tilde{g}_{j\bar{k}}dz_j \wedge d\overline{z}_k,$$

and let $a_i = \dfrac{\partial z_1}{\partial z_i}$, $i = 2, \cdots, n$.Then by (3.3) and (3.4), we have

$$\begin{aligned}\tilde{g}_{j\bar{k}} \quad &= \frac{\delta_{jk}}{1+|z|^2} - \frac{z_k\overline{z}_j}{(1+|z|^2)^2} - \frac{z_1\overline{z}_j\overline{a}_j}{(1+|z|^2)^2} \\ &- \frac{z_k\overline{z}_1 a_j}{(1+|z|^2)^2} + \frac{a_j\overline{a}_k}{1+|z|^2} - \frac{|z_1|^2 a_j\overline{a}_k}{(1+|z|^2)^2},\end{aligned}$$

for $j, k = 2, \cdots, n$. Since the Ricci tensor is given by

$$R_{l\overline{m}} = -\frac{\partial^2 \log det(\tilde{g}_{j\bar{k}})}{\partial z_l \partial \overline{z}_m}, l, m = 2, \cdots, n.$$

So its Ricci curvature form is

$$Ric(\omega) = \frac{i}{2\pi}\sum_{l,m=2}^{n} R_{l\overline{m}} dz_l \wedge d\overline{z}_m = -\frac{i}{2\pi}\partial\overline{\partial}\log det(\tilde{g}_{j\bar{k}}).$$

Now, we need to compute the determinant $det(\tilde{g}_{j\bar{k}})$. In order to do this, we let

$$K_{j\bar{k}} = \delta_{jk} + a_j\overline{a}_k - \frac{1}{1+|z|^2}(\overline{z}_j + \overline{z}_1 a_j)\overline{(\overline{z}_k + \overline{z}_1 a_k)}.$$

Then

$$\tilde{g}_{j\bar{k}} = \frac{1}{1+|z|^2}K_{j\bar{k}},\ j, k = 2, \cdots, n. \tag{3.5}$$

The matrix $K = (K_{i\bar{j}})$ can be represented by

$$K = I + AA^H - \frac{1}{1+|z|^2}BB^H,$$

where

$$\begin{aligned} A \quad &= (a_2, \cdots, a_n)^T, \\ B \quad &= (\overline{z}_2 + \overline{z}_1 a_2, \cdots, \overline{z}_n + \overline{z}_1 a_n)^T. \end{aligned}$$

By the notation above, we get

$$\begin{aligned}
KA &= (I + AA^H - \frac{1}{1+|z|^2}BB^H)A \\
&= A + (AA^H)A - \frac{1}{1+|z|^2}(BB^H)A \\
&= A + (A^HA)A - \frac{1}{1+|z|^2}(B^HA)B \\
&= (1+|a|^2)A - \frac{1}{1+|z|^2}(B^HA)B.
\end{aligned}$$

$$\begin{aligned}
KB &= (I + AA^H - \frac{1}{1+|z|^2}BB^H)B \\
&= B + (AA^H)B - \frac{1}{1+|z|^2}(BB^H)B \\
&= (A^HB)A + B - \frac{1}{1+|z|^2}(B^HB)B \\
&= (A^HB)A + (1 - \frac{|B|^2}{1+|z|^2})B.
\end{aligned}$$

Hence, the vector subspace $span\{A, B\}$ is $K-invariant$. Furthermore, K is identity on the complement of $span\{A, B\}$. So we have

$$\begin{aligned}
detK = &\quad (1+|a|^2)(1-\frac{|B|^2}{1+|z|^2}) + \frac{1}{1+|z|^2}|B^HA|^2 \\
= &\quad \frac{1}{1+|z|^2}\big((1+|a|^2)(1+|z|^2) - (1+|a|^2)|B|^2 + |B^HA|^2\big) \\
= &\quad \frac{1}{1+|z|^2}\big(1+|a|^2+|z|^2+|z|^2|a|^2-|B|^2-|a|^2|B|^2+\big|\sum_{i=1}^{n} a_iz_i + z_1|a|^2\big|^2\big) \\
= &\quad \frac{1}{1+|z|^2}\big\{1+|a|^2+|z|^2+|z|^2|a|^2 \\
&\quad -(|z'|^2+|z_1|^2|a|^2+\sum_{i=2}^{n} a_iz_i\overline{z}_1+\sum_{i=2}^{n}\overline{a}_i\overline{z}_iz_1)+\big|\sum_{i=2}^{n} a_iz_i + z_1|a|^2\big|^2 \\
&\quad -|a|^2(|z'|^2+|z_1|^2|a|^2+\sum_{i=2}^{n} a_iz_i\overline{z}_1+\sum_{i=2}^{n}\overline{a}_i\overline{z}_iz_1)\big\} \\
= &\quad \frac{1}{1+|z|^2}(1+|a|^2+|z_1|^2+\sum_{i=2}^{n} a_iz_i\overline{z}_1+\sum_{i=2}^{n}\overline{a}_i\overline{z}_iz_1+|\sum_{i=2}^{n} a_iz_i|^2) \\
= &\quad \frac{1}{1+|z|^2}(1+|a|^2+|\sum_{i=2}^{n} a_iz_i - z_1|^2),
\end{aligned} \tag{3.6}$$

where $z' = (z_2, \cdots, z_n)$. Let f be the defining function of M on U_0, i.e.

$$f = F(1, \frac{Z_1}{Z_0}, \cdots, \frac{Z_n}{Z_0}) = \frac{F}{Z_0^d}.$$

Then

$$a_i = \frac{\partial z_1}{\partial z_i} = -\frac{\frac{\partial f}{\partial z_i}}{\frac{\partial f}{\partial z_1}} = -\frac{F_i}{F_1}, \; i = 2, \cdots, n, \tag{3.7}$$

where we define $F_i = \dfrac{\partial F}{\partial Z_i}$ for $i = 0, \cdots, n$. Then by the homogeneity of F, we have

$$\begin{aligned} (\sum_{i=2}^{n} a_i z_i) - z_1 &= -(\sum_{i=2}^{n} \frac{F_i}{F_1}\frac{Z_i}{Z_0}) - \frac{Z_1}{Z_0} \\ &= -\frac{1}{F_1 Z_0}(\sum_{i=1}^{n} Z_i F_i) = \frac{F_0}{F_1}. \end{aligned} \tag{3.8}$$

on M. By (3.5), (3.6), and (3.8), we have

$$\begin{aligned} det\tilde{g}_{j\bar{k}} &= \frac{1}{1+|z|^2} det K_{j\bar{k}} \\ &= \frac{1}{(1+|z|^2)^n}\frac{1}{|F_1|^2}(\sum_{i=0}^{n}|F_i|^2). \end{aligned} \tag{3.9}$$

Then by (3.1), we get

$$\begin{aligned} det\tilde{g}_{j\bar{k}} &= \frac{1}{(1+|z|^2)^{n-d+1}}\frac{1}{|F_1|^2}\frac{\sum_{i=0}^{n}|F_i|^2}{(1+|z|^2)^{d-1}} \\ &= \frac{1}{(1+|z|^2)^{n-d=1}}\frac{1}{|\frac{\partial f}{\partial z_1}|^2 |Z_0|^{2(d-1)}}\frac{|Z_0|^{2(d-1)}|\nabla F|^2}{(\sum_{i=0}^{n}|Z_i|^2)^{(d-1)}} \\ &= \frac{1}{(1+|z|^2)^{n-d=1}}\frac{1}{|\frac{\partial f}{\partial z_1}|^2} e^{\varphi}. \end{aligned}$$

The conclution follows from the formula of the Ricci curvature and the above equation. ❑

In order to represent the K energy in terms of the polynomial F, we need the following purely algebraic lemma:

Lemma 3.2 *With the same notations as above, let η be a (1,1) form on $\mathbb{CP}^n$. Let $\pi : \mathbb{C}^{n+1} \to \mathbb{CP}^n$ be the projection. Let*

$$\pi^*\eta = \frac{i}{2\pi}\sum_{j,k=1}^{\infty} \tilde{a}_{j\bar{k}} dZ_j \wedge d\overline{Z}_k. \tag{3.10}$$

Then on M,

$$\eta \wedge \omega^{n-2} = \frac{|Z|^2}{n-1}\Big(\sum_{j=0}^{n} \tilde{a}_{j\bar{j}} - \frac{\sum_{j,k=0}^{n} \tilde{a}_{j\bar{k}F_k\overline{F}_j}}{|\nabla F|^2}\Big)\omega^{n-1} \tag{3.11}$$

for $|Z|^2 = \sum_{i=0}^{n} |Z_i|^2$.

Proof. As the proof of Lemma 3.1, we just have to deal with the problem on $U_0 \cap \{\frac{\partial F}{\partial Z_1} \neq 0\}$, where $U_0 = \{[Z_0, \cdots, Z_n]\big|\, |Z_0| > \frac{1}{2}|Z_j|, j = 1, 2, \cdots, n\}$ in $\mathbb{CP}^n$. Note that $\tilde{a}_{i\bar{j}}$, $i, j = 0, \cdots, n$, are homogeneous functions of order (-2), so (3.11) is well defined. Define $A_{j\bar{k}}$ on $\mathbb{CP}^n$ as follows:

$$\begin{aligned} &\eta \wedge \omega^{n-2} = \Big(\frac{i}{2\pi}\Big)^{n-1}(-1)^{\frac{1}{2}(n-1)(n-2)} \\ &\cdot \sum_{j,k=1}^{n} (-1)^{j+k} A_{j\bar{k}} dz_1 \wedge \cdots \wedge \hat{dz_j} \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge \hat{d\overline{z}_k} \wedge \cdots \wedge d\overline{z}_n. \end{aligned} \tag{3.12}$$

where the symbol "$\hat{}$" over dz_j and $d\overline{z}_k$ means these two differential forms are deleted from the expression. Define $b = (b_1, \cdots, b_n)$ by

$$b = (1, -a_2, \cdots, -a_n) = \Big(1, -\frac{\partial z_1}{\partial z_2}, \cdots, -\frac{\partial z_1}{\partial z_n}\Big) = \Big(1, \frac{F_2}{F_1}, \cdots, \frac{F_n}{F_1}\Big).$$

Then the equation (3.12) can be represented by

$$\begin{aligned} &\eta \wedge \omega^{n-2} = \Big(\frac{i}{2\pi}\Big)^{n-1}(-1)^{\frac{1}{2}(n-1)(n-2)} \\ &\cdot \sum_{j,k=1}^{n} A_{j\bar{k}} b_j \overline{b}_k dz_2 \wedge \cdots \wedge dz_n \wedge d\overline{z}_2 \wedge \cdots \wedge d\overline{z}_n \end{aligned} \tag{3.13}$$

on M. Let

$$\eta = \frac{i}{2\pi}\sum_{l,m=1}^{n} a_{l\overline{m}} dz_l \wedge d\overline{z}_m, \tag{3.14}$$

and fix s, t. By (3.12), we have

$$\begin{aligned} &\frac{i}{2\pi} dz_s \wedge d\overline{z}_t \wedge \eta \wedge \omega^{n-2} \\ = \;& \frac{i}{2\pi} dz_s \wedge d\overline{z}_t \wedge \frac{i}{2\pi}\sum_{l,m=1}^{n} a_{l\overline{m}} dz_l \wedge d\overline{z}_m \wedge \omega^{n-2} \\ = \;& \Big(\frac{i}{2\pi}\Big)^n (-1)^{\frac{1}{2}(n-1)(n-2)} (-1)^{n-1} A_{s\bar{t}} dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n. \end{aligned} \tag{3.15}$$

We also have the following fact :

$$\begin{aligned} & \frac{i}{2\pi}dz_s \wedge d\overline{z}_t \wedge \frac{i}{2\pi}\sum_{l,m=1}^{n} a_{l\overline{m}}dz_l \wedge d\overline{z}_m \wedge \omega^{n-2} \\ = \ & \frac{1}{n(n-1)}\big(\sum_{\alpha,\beta=1}^{n}(g^{\alpha\overline{\beta}}a_{\alpha\overline{\beta}})g^{s\overline{t}} - \sum_{\alpha,\beta=1}^{n} g^{\alpha\overline{t}}g^{s\overline{\beta}}a_{\alpha\overline{\beta}}\big)\omega. \end{aligned} \tag{3.16}$$

By (3.3), we have

$$\omega^n = (\frac{i}{2\pi})^n(-1)^{n(n-1)}\frac{n!}{(1+|z|^2)^{n+1}}dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n. \tag{3.17}$$

By (3.15), (3.16), and (3.17), we get

$$\begin{aligned} & (\frac{i}{2\pi})^n(-1)^{\frac{1}{2}n(n-1)}A_{s\overline{t}}dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n \\ = \ & (\frac{i}{2\pi})^n(-1)^{\frac{1}{2}n(n-1)}\frac{(n-2)!}{(1+|z|^2)^{n+1}}dz_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_n \\ & \cdot\big(\sum_{\alpha,\beta=1}^{n}(g^{\alpha\overline{\beta}}a_{\alpha\overline{\beta}})g^{s\overline{t}} - \sum_{\alpha,\beta=1}^{n} g^{\alpha\overline{t}}g^{s\overline{\beta}}a_{\alpha\overline{\beta}}\big). \end{aligned}$$

So

$$A_{s\overline{t}} = \frac{(n-2)!}{(1+|z|^2)^{n+1}}\big(\sum_{\alpha,\beta=1}^{n}(g^{\alpha\overline{\beta}}a_{\alpha\overline{\beta}})g^{s\overline{t}} - \sum_{\alpha,\beta=1}^{n} g^{\alpha\overline{t}}g^{s\overline{\beta}}a_{\alpha\overline{\beta}}\big), \tag{3.18}$$

for $s, t = 1, \cdots, n$. By (3.18), we have

$$\begin{aligned} & \sum_{j,k=1}^{n} A_{j\overline{k}}b_j\overline{b}_k = \frac{(n-2)!}{(1+|z|^2)^{n+1}} \\ & \cdot\big(\sum_{\alpha,\beta=1}^{n} g^{\alpha\overline{\beta}}a_{\alpha\overline{\beta}}\sum_{j,k=1}^{n} g^{j\overline{k}}b_j\overline{b}_k - \sum_{j,k,\alpha,\beta=1}^{n} g^{\alpha\overline{k}}g^{j\overline{\beta}}a_{\alpha\overline{\beta}}\big). \end{aligned} \tag{3.19}$$

Now, we need to deal with the right hand side of (3.19).

From (3.10) and (3.14), we have

$$\begin{cases} a_{l\overline{m}} = \tilde{a}_{l\overline{m}}|Z_0|^2, & l, m \neq 0; \\ \sum_{j=1}^{n} z_j a_{j\overline{m}} = -\tilde{a}_{0\overline{m}}|Z_0|^2, & m \neq 0; \\ \sum_{k=1}^{n} z_k a_{l\overline{k}} = -\tilde{a}_{l\overline{0}}|Z_0|^2, & l \neq 0; \\ \sum_{j,k=1}^{n} z_j \overline{z}_k a_{j\overline{k}} = \tilde{a}_{0\overline{0}}|Z_0|^2. \end{cases} \tag{3.20}$$

Since $g^{\alpha\overline{\beta}} = (1+|z|^2)(\delta_{\alpha\beta} + z_\alpha \overline{z}_\beta)$, from (3.20), we have

$$\begin{aligned} \sum_{\alpha,\beta=1}^{n} g^{\alpha\overline{\beta}} a_{\alpha\overline{\beta}} &= (1+|z|^2) \sum_{\alpha,\beta=1}^{n} (\delta_{\alpha\beta} + z_\alpha \overline{z}_\beta) a_{\alpha\overline{\beta}} \\ &= (1+|z|^2)(\sum_{\alpha,\beta=1}^{n} \delta_{\alpha\beta} a_{\alpha\overline{\beta}} + \sum_{\alpha,\beta=1}^{n} z_\alpha \overline{z}_\beta a_{\alpha\overline{\beta}}) \\ &= (1+|z|^2)(\sum_{\alpha=1}^{n} \tilde{a}_{\alpha\overline{\alpha}}|Z_0|^2 + \tilde{a}_{0\overline{0}}|Z_0|^2) \\ &= |Z_0|^2(1+|z|^2)\sum_{i=0}^{n} \tilde{a}_{i\overline{i}}. \end{aligned} \tag{3.21}$$

By (3.8), we have

$$\sum_{i=1}^{n} z_i b_i = z_i - \sum_{i=2}^{n} z_i a_i = -\frac{F_0}{F_1}$$

on M. By (3.7), (3.20) and the equation above

$$\begin{aligned} \sum_{j,k=1}^{n} g^{j\overline{k}} b_j \overline{b}_k &= (1+|z|^2) \sum_{j,k=1}^{n} (\delta_{jk} + z_j \overline{z}_k) b_j \overline{b}_k \\ &= (1+|z|^2)(\sum_{j=1}^{n} b_j \overline{b}_j + \sum_{j,k=1}^{n} z_j \overline{z}_k b_j \overline{b}_k) \\ &= (1+|z|^2)\frac{\sum_{i=0}^{n}|F_i|^2}{|F_1|^2} \\ &= (1+|z|^2)\frac{|\nabla F|^2}{|F_1|^2}. \end{aligned} \tag{3.22}$$

$$\begin{aligned}
&\sum_{j,k,\alpha,\beta=1}^{n} g^{\alpha\bar{k}} g^{j\bar{\beta}} a_{\alpha\bar{\beta}} b_j \bar{b}_k \\
&= (1+|z|^2)^2 \sum_{j,k,\alpha,\beta=1}^{n} (\delta_{\alpha k} + z_\alpha \bar{z}_k)(\delta_{j\beta} + z_j \bar{z}_\beta) a_{\alpha\bar{\beta}} b_j \bar{b}_k \qquad (3.23)\\
&= |Z_0|^2 (1+|z|^2) \frac{\sum_{\alpha,\beta=0}^{n} \tilde{a}_{\alpha\bar{\beta}} \overline{F}_\alpha F_\beta}{|F_1|^2}.
\end{aligned}$$

By (3.21), (3.22) and (3.23), we have

$$\begin{aligned}
&\sum_{\alpha,\beta=1}^{n} g^{\alpha\bar{\beta}} a_{\alpha\bar{\beta}} \sum_{j,k=1}^{n} g^{j\bar{k}} b_j \bar{b}_k - \sum_{j,k,\alpha,\beta=1}^{n} g^{\alpha\bar{k}} g^{j\bar{\beta}} a_{\alpha\bar{\beta}} \\
&= |Z_0|^2 (1+|z|^2)^2 \frac{|\nabla F|^2 \sum_{j=0}^{n} \tilde{a}_{j\bar{j}}}{|F_1|^2} - |Z_0|^2 (1+|z|^2)^2 \frac{\sum_{j,k=0}^{n} \tilde{a}_{j\bar{k}} \overline{F}_j F_k}{|F_1|^2} \qquad (3.24)\\
&= |Z_0|^2 (1+|z|^2)^2 \frac{|\nabla F|^2}{|F_1|^2} \Big(\sum_{j=0}^{n} \tilde{a}_{j\bar{j}} - \sum_{j,k=0}^{n} \frac{\tilde{a}_{j\bar{k}} \overline{F}_j F_k}{|\nabla F|^2}\Big).
\end{aligned}$$

Hence the expression (3.13) can be replaced by

$$\begin{aligned}
\eta \wedge \omega^{n-2} = \; & \Big(\frac{i}{2\pi}\Big)^{n-1} (-1)^{\frac{1}{2}(n-1)(n-2)} |Z_0|^2 \frac{1}{(1+|z|^2)^{n-1}} (n-2)! \frac{|\nabla F|^2}{|F_1|^2} \qquad (3.25)\\
& \cdot \Big(\sum_{j=0}^{n} \tilde{a}_{j\bar{j}} - \sum_{j,k=0}^{n} \frac{\tilde{a}_{j\bar{k}} \overline{F}_j F_k}{|\nabla F|^2}\Big) dz_2 \wedge \cdots \wedge dz_n \wedge d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_n.
\end{aligned}$$

By (3.3) and (3.9), we have

$$\begin{aligned}
\omega^{n-1} = \; & \Big(\frac{i}{2\pi}\Big)^{n-1} (n-1)! (-1)^{\frac{1}{2}(n-1)(n-2)} \frac{1}{(1+|z|^2)^n} \frac{|\nabla F|^2}{|F_1|^2} \qquad (3.26)\\
& \cdot dz_2 \wedge \cdots \wedge dz_n \wedge d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_n.
\end{aligned}$$

By (3.25) and (3.26), we get

$$\begin{aligned}
\eta \wedge \omega^{n-2} &= \frac{1}{n-1} \frac{1}{1+|z|^2} |Z_0|^2 \Big(\sum_{i=0}^{n} \tilde{a}_{i\bar{i}} - \sum_{j,k=0}^{n} \frac{\tilde{a}_{j\bar{k}} F_k \overline{F}_j}{|\nabla F|^2}\Big) \omega^{n-1} \\
&= \frac{1}{n-1} \frac{\sum_{i=0}^{n} |Z_i|^2}{|Z_0|^2} |Z_0|^2 \Big(\sum_{i=0}^{n} \tilde{a}_{i\bar{i}} - \sum_{j,k=0}^{n} \frac{\tilde{a}_{j\bar{k}} F_k \overline{F}_j}{|\nabla F|^2}\Big) \omega^{n-1} \\
&= \frac{|Z|^2}{n-1} \Big(\sum_{i=0}^{n} \tilde{a}_{i\bar{i}} - \sum_{j,k=0}^{n} \frac{\tilde{a}_{j\bar{k}} F_k \overline{F}_j}{|\nabla F|^2}\Big) \omega^{n-1},
\end{aligned}$$

where $|Z|^2 = \sum_{i=0}^{n} |Z_i|^2$. So we complete the proof. ❑

Lemma 3.3 *Let φ be the function defined in (3.1) and let*

$$\theta = -\frac{\sum_{j=0}^n \lambda_j |Z_j|^2}{\sum_{j=0}^n |Z_j|^2} = -\frac{\sum_{j=0}^n \lambda_j |Z_j|^2}{|Z|^2}.$$

Then we have

$$\begin{aligned} & \frac{i}{2\pi}\partial\varphi \wedge \overline{\partial}\theta \wedge \omega^{n-2} \\ = \ & \frac{1}{n-1}\Big(-\sum_{j=0}^n \big(\frac{XF}{|\nabla F|^2}\big)_j \overline{F}_j + \frac{\sum_{j=0}^n \lambda_j |F_j|^2}{|\nabla F|^2} - (d-1)\theta \Big)\omega^{n-1}. \end{aligned} \tag{3.27}$$

Furthermore, we have

$$\begin{aligned} & \frac{i}{2\pi}\int_M \partial\varphi \wedge \overline{\partial}\theta \wedge \omega^{n-2} \\ = \ & -\frac{1}{n-1}\int_M \sum_{i=0}^n \big(\frac{XF}{|\nabla F|^2}\big)_j \overline{F}_j \omega^{n-1} + \frac{n-d+1}{n-1}\int_M \theta\omega^{n-1}. \end{aligned} \tag{3.28}$$

Proof. Let $\eta = \frac{i}{2\pi}\partial\varphi \wedge \overline{\partial}\theta$ be a (1,1) form on $\mathbb{CP}^n$. Let $\pi : \mathbb{C}^{n+1} \to \mathbb{CP}^n$ be the projection, and let

$$\pi^*\eta = \frac{i}{2\pi}\sum_{j,k=0}^n \tilde{a}_{j\bar{k}} dZ_j \wedge d\overline{Z}_k$$

as in (3.10). Then we have

$$\tilde{a}_{j\bar{k}} = \frac{\partial\varphi}{\partial Z_j}\frac{\partial\theta}{\partial \overline{Z}_k}.$$

By the equation above, we have

$$\sum_{j=0}^n \tilde{a}_{j\bar{j}} = \frac{\sum_{j=0}^n \lambda_j |F_j|^2 - \sum_{j=0}^n \big(\sum_{i=0}^n \lambda_i Z_i \frac{\partial F}{\partial Z_i}\big)_j \overline{F}_j}{|Z|^2 |\nabla F|^2} - (d-1)\frac{\theta}{|Z|^2}$$

and

$$\frac{\sum_{j,k=0}^n \tilde{a}_{j\bar{k}} \overline{F}_j F_k}{|\nabla F|^2} = -\frac{\sum_{i=0}^n \lambda_i Z_i \frac{\partial F}{\partial Z_i} \sum_{j,k=0}^n F_{jk}\overline{F}_j \overline{F}_k}{|Z|^2 |\nabla F|^4}$$

on M. By Lemma 3.2, we get (3.27)

$$
\begin{aligned}
&\frac{i}{2\pi}\partial\varphi\wedge\overline{\partial}\theta\wedge\omega^{n-2} = \eta\wedge\omega^{n-2}\\
&= \frac{|Z|^2}{n-1}\Big(\sum_{j=0}^{n}\tilde{a}_{j\bar{j}} - \frac{\sum_{j,k=0}^{n}\tilde{a}_{j\bar{k}}\overline{F}_j F_k}{|\nabla F|^2}\Big)\omega^{n-1}\\
&= \frac{1}{n-1}\Big(\frac{\sum_{j=0}^{n}\lambda_j|F_j|^2 - \sum_{j=0}^{n}(XF)_j\overline{F}_j}{|\nabla F|^2} - (d-1)\theta + \frac{XF\sum_{j,k=0}^{n}F_{jk}\overline{F}_j\overline{F}_k}{|\nabla F|^4}\Big)\omega^{n-1}\\
&= \frac{1}{n-1}\Big(\frac{\sum_{j=0}^{n}\lambda_j|F_j|^2}{|\nabla F|^2} - (d-1)\theta - \Big(\frac{\sum_{j=0}^{n}(XF)_j\overline{F}_j|\nabla F|^2 - XF\sum_{j,k=0}^{n}F_{jk}\overline{F}_j\overline{F}_k}{|\nabla F|^4}\Big)\Big)\omega^{n-1}\\
&= \frac{1}{n-1}\Big(\frac{\sum_{j=0}^{n}\lambda_j|F_j|^2}{|\nabla F|^2} - (d-1)\theta - \sum_{j=0}^{n}\Big(\frac{XF}{|\nabla F|^2}\Big)_j\overline{F}_j\Big)\omega^{n-1}.
\end{aligned}
$$

Now, let $\eta = \frac{i}{2\pi}\partial\overline{\partial}\theta$. By Lemma 3.2, we have

$$\frac{i}{2\pi}\partial\overline{\partial}\theta\wedge\omega^{n-2} = \frac{1}{n-1}\Big(\frac{\sum_{j=0}^{n}\lambda_j|F_j|^2}{|\nabla F|^2} - n\theta\Big)\omega^{n-1}. \tag{3.29}$$

By(3.27), (3.29) and the Stokes Theorem, we get (3.28)

$$
\begin{aligned}
&\frac{i}{2\pi}\int_M \partial\varphi\wedge\overline{\partial}\theta\wedge\omega^{n-2}\\
= &-\frac{1}{n-1}\Big(\int_M\sum_{j=0}^{n}\Big(\frac{XF}{|\nabla F|^2}\Big)_j\overline{F}_j\omega^{n-1} + (d-1)\int_M\theta\omega^{n-1} + \int_M(-n)\omega^{n-1}\Big)\\
= &-\frac{1}{n-1}\Big(\int_M\sum_{j=0}^{n}\Big(\frac{XF}{|\nabla F|^2}\Big)_j\overline{F}_j\omega^{n-1} - (n-d+1)\int_M\theta\omega^{n-1}\Big).
\end{aligned}
$$

❑

Theorem 3.4 *The K energy $\mathcal{M}(t)$ can be represented as*

$$
\begin{aligned}
\mathcal{M}(t) = \frac{2}{d}\int_1^t\Big(\int_{M_r}\frac{1}{r}\Big(-\sum_{j=0}^{n}\Big(\frac{XF_r}{|\nabla F|^2}\Big)_j\overline{(F_r)_j}\omega^{n-1}\\
+(n-d+1)\theta\omega^{n-1}\Big)\Big)dr,
\end{aligned}
\tag{3.30}
$$

where

$$F_r(Z_0,\cdots,Z_n) = F(r^{-\lambda_0}Z_0,\cdots,r^{-\lambda_n}Z_n)$$

and M_r is the zero set of $F_r = 0$. In particular, we have

$$
\begin{aligned}
&t\frac{d}{dt}\mathcal{M}(t) = \frac{2}{d}\Big(- \int_{M_t} \sum_{j=0}^{n} \big(\frac{XF_t}{|\nabla F|^2}\big)_j \overline{(F_t)_j}\omega^{n-1} \\
&+(n-d+1)\int_{M_t} \theta\omega^{n-1}\Big).
\end{aligned}
\tag{3.31}
$$

Proof. By Proposition 1.5, we have

$$
\begin{aligned}
\mathcal{M}(t) \quad &= \frac{2(n-1)}{d}\int_1^t \Big(\int_{M_r} \frac{1}{r}(Ric(\omega|_{M_r}) - (n-d+1)\omega|_{M_r})\theta\omega^{n-2}\Big)dr \\
&= \frac{2(n-1)}{d}\int_1^t \Big(\int_{M_r} \frac{1}{r}(\frac{i}{2\pi}\partial\overline{\partial}\varphi)\theta\omega^{n-2}\Big)dr \\
&= \frac{2(n-1)}{d}\int_1^t \Big(\int_{M_r} \frac{1}{r}(-\frac{1}{n-1}\sum_{j=0}^{n}\big(\frac{XF_r}{|\nabla F|^2}\big)_j \overline{(F_r)_j}\omega^{n-1} \\
&+\frac{n-d+1}{n-1}\theta\omega^{n-1})\Big)dr \\
&= \frac{2}{d}\int_1^t \Big(\int_{M_r} \frac{1}{r}\Big(-\sum_{j=0}^{n}\big(\frac{XF_r}{|\nabla F|^2}\big)_j \overline{(F_r)_j}\omega^{n-1} \\
&+(n-d+1)\theta\omega^{n-1}\Big)\Big)dr.
\end{aligned}
$$

By the fundamental theorem of calculus, we get

$$
\begin{aligned}
t\frac{d}{dt}\mathcal{M}(t) \quad &= t\frac{2}{d}\int_{M_t}\Big(-\frac{1}{t}\sum_{j=0}^{n}\big(\frac{XF_t}{|\nabla F|^2}\big)_j \overline{(F_t)_j}\omega^{n-1} \\
&+(n-d+1)\theta\omega^{n-1}\Big) \\
&= \frac{2}{d}\Big(-\int_{M_t}\sum_{j=0}^{n}\big(\frac{XF_t}{|\nabla F|^2}\big)_j \overline{(F_t)_j}\omega^{n-1} \\
&+(n-d+1)\int_{M_t}\theta\omega^{n-1}\Big).
\end{aligned}
$$

❑

4 The limit of the derivative of the K energy

In last section, we get a explicit formula of $t\dfrac{d}{dt}\mathcal{M}(t)$. Here, we are going to compute the limit $\lim\limits_{t\to 0} t\dfrac{d}{dt}\mathcal{M}(t)$ in Theorem 3.4. In order to do this, we need some combinatoric techniques first.

Let $(\delta_i, \sigma_i), i = 0, \cdots, p$, be a sequence of pair of nonnegative rational numbers. Let $\delta_0 = 0$. We assume that the sequence is "generic" in the sence that

1. All δ_i, $i = 0, \cdots, p$, are distinct numbers. Hence, each $\delta_i, i = 1, \cdots, p$, is positive rational number.

2. Define the line $\xi_i(x) = \delta_i + \sigma_i x, i = 0, \cdots, p$. None of three such lines intersect at the same point.

Now, suppose (δ_i, σ_i), $i = 0, \cdots, p$, are generic, define (i_k, r_k), $k = 0, \cdots, m$, by induction as follows: let $i_0 = 0$, $r_0 = 0$. If (i_k, r_k) has been defined, then

1. If for any $r > r_k$

   $$\delta_{i_k} + \sigma_{i_k} r < \delta_i + \sigma_i r,\ i \neq i_k,$$

   then let $m = k$ and stop.

2. If not, then define i_{k+1} and $r_{k+1} > r_k$ such that

   $$\delta_{i_k} + \sigma_{i_k} r_{k+1} = \delta_{i_{k+1}} + \sigma_{i_{k+1}} r_{k+1} \leq \delta_i + \sigma_i r_{k+1}, \tag{4.1}$$

   where $i = 1, \cdots, p$. Note that (i_k, r_k), $k = 0, \cdots, m$, are unique definite since (δ_i, σ_i), $i = 0, \cdots, p$, are generic.

From the process above, we have the obvious.

Remark. (i_k, r_k), $k = 0, 1, \cdots$, is a finite sequence. In particular, the sequence stop at (i_m, r_m). Indeed, by the construction of $i_k{}'s$, we have

$$\sigma_{i_0} > \sigma_{i_1} > \cdots > \sigma_{i_k} > \cdots .$$

Hence all $i_k{}'s$ must be distinct. Since $0 \le i_k \le p$, we have at most $p+1$ distinct $i_k{}'s$. The second statement follows from the first item of the construction above.

Let

$$\xi(x) = \min_{i\ge 0}(\delta_i + \sigma_i x). \tag{4.2}$$

The function $\xi(x)$ is a piecewise linear function, which be non-smooth at r_k, $k = 1, \cdots, m$. And the function $\xi(x)$ is differentiable almost everywhere.

Lemma 4.1 *Assume that $\sigma_{i_m} = 0$, we have*

$$\sum_{k=0}^{m-1}(-\delta_{i_k} + \delta_{i_{k+1}})(\sigma_{i_k} + \sigma_{i_{k+1}} - 1) = \int_0^\infty \xi'(x)(\xi'(x) - 1)dx. \tag{4.3}$$

Proof. Note that $\xi \equiv \delta_{i_m}$ is a constant function if x large enough.

$$\int_0^\infty \xi'(x)dx = \lim_{b\to\infty}\int_0^b \xi'(x)dx = \lim_{b\to\infty}(\xi(b) - \xi(0)) = \delta_{i_m} - \delta_{i_0}.$$

Using the summation by parts, we have

$$\int_0^\infty \xi'(x)^2 dx = r_1(\sigma_{i_0})^2 + \sum_{k=1}^{m-1}\sigma_{i_k}^2(r_{k+1} - r_k) = \sum_{k=0}^{m-1} r_{k+1}(\sigma_{i_k}^2 - \sigma_{i_{k+1}}^2)$$

By definition of $r_k, k = 0, \cdots, m$, in (4.1), we have

$$-\delta_{i_k} + \delta_{i_{k+1}} = (\sigma_{i_k} - \sigma_{i_{k+1}})r_{k+1}, \text{ for } k = 0, \cdots, m-1.$$

Thus we have

$$\begin{aligned}
& \sum_{k=0}^{m-1}(-\delta_{i_k} + \delta_{i_{k+1}})(\sigma_{i_k} + \sigma_{i_{k+1}} - 1) \\
= & \sum_{k=0}^{m-1} r_{k+1}(\sigma_{i_k}^2 - \sigma_{i_{k+1}}^2) - (\delta_{i_m} - \delta_{i_0}) \\
= & \int_0^\infty \xi'(x)(\xi'(x) - 1)dx.
\end{aligned}$$

❑

Consider the smooth hypersurface $M \subset \mathbb{CP}^n$ defined by the polynomial $F = 0$ of degree d. Let $X = \sum_{i=0}^{n} \lambda_i Z_i \dfrac{\partial}{\partial Z_i}$ be the vector field for integers $(\lambda_0, \cdots, \lambda_n)$ such that $\sum_{i=0}^{n} \lambda_i = 0$. Let M_t be defined by the equation

$$F_t(Z_0, \cdots, Z_n) = F(t^{-\lambda_0} Z_0, \cdots, t^{-\lambda_n} Z_n). \tag{4.4}$$

We write F_t as

$$F_t(Z_0, \cdots, Z_n) = t^{\delta} \sum_{i=0}^{p} a_i t^{\delta_i} Z_0^{\alpha_0^{(i)}} \cdots Z_n^{\alpha_n^{(i)}}, \tag{4.5}$$

where $\delta_0 = 0$, and $\delta_i \geq 0, i = 1, \cdots, p$. And

$$\delta = -\lambda = \min_{0 \leq i \leq p} \big(\sum_{k=0}^{n} (-\lambda_k) \cdot \alpha_k^{(i)} \big).$$

By (4.4), we have

$$X(Z_0^{\alpha_0^{(i)}} \cdots Z_n^{\alpha_n^{(i)}}) = -(\delta_i + \delta) Z_0^{\alpha_0^{(i)}} \cdots Z_n^{\alpha_n^{(i)}} \tag{4.6}$$

for $i = 0, \cdots, p$.

The sequence $(\delta_i, \alpha_k^{(i)}), i = 0, \cdots, p, k = 0, \cdots, n$, be the pair of nonnegative rational numbers which satisfies

1. All $\delta_i, i = 0, \cdots, p$, are distinct;
2. None of the three lines defined by $\delta_i + \alpha_k^{(i)} x$ for $i = 0, \cdots, p$, intersect at the same point, where $k = 0, \cdots, n$.

So we may assume that the choice of $(\lambda_0, \cdots, \lambda_n)$ is generic. Without loss of generality, we may assume that $a_0 = 1$, and $0 = \delta_0 < \delta_1 < \cdots < \delta_p$. We also assume that $a_1, \cdots, a_p$ are all nonzero. Moreover, since M is smooth, we see that for each $0 \leq k \leq n$, there is an $0 \leq i \leq p$ such that $\alpha_k^{(i)} = 0$.

Let

$$U_i = \{[Z_0, \cdots, Z_n] \in \mathbb{CP}^n \big| \, |Z_i| > \frac{1}{2} |Z_j|, \, j = 0, \cdots, n\}.$$

Then $\cup U_i = \mathbb{CP}^n$. Let $P_i = \{Z_i = 0\}$ and $P_{ij} = P_i \cap P_j$ for $i \neq j$ and $i, j = 0, \cdots, n$. Let $\sigma > 0$ be chosen so that $\sigma < \frac{1}{d} \min_{i \geq 1}(\delta_i)$. Let $d(\cdot\,,\cdot)$ be the distance induced by the Fubini-Study metric ω on $\mathbb{CP}^n$, and define

$$V_{ij}^t = \{z \in \mathbb{CP}^n \big| \, d(z\,, P_{ij}) < |t|^\sigma\},\ i \neq j,\ i, j = 0, \cdots, n.$$

By (4.5), we have $t^{-\delta} F_t \to Z_0^{\alpha_0^{(0)}} \cdots Z_n^{\alpha_n^{(0)}}$ as $t \to 0$. Intuitively, M_t goes to the hyperplanes defined by $Z_0^{\alpha_0^{(0)}} \cdots Z_n^{\alpha_n^{(0)}} = 0$.

Lemma 4.2 *There is a $\sigma_1 > \sigma$ such that for any $0 \leq k \leq n$ and*

$$[Z_0, \cdots, Z_n] \in (M_t - \cup_{i,j=0}^n V_{ij}^t) \cap U_k,$$

one can find a unique $l \neq k$ such that

$$|\frac{Z_l}{Z_k}| < |t|^{\sigma_1}$$

for t small enough, where $[Z_0, \cdots, Z_n] \in M_t$.

Proof. Since $[Z_0, \cdots, Z_n] \in U_k$, we have $|Z_j| < 2|Z_k|$, $j = 0, \cdots, n$. By (4.5) we have

$$|Z_0^{\alpha_0^{(0)}} \cdots Z_n^{\alpha_n^{(0)}}| \leq 2^d \sum_{i=1}^p a_i |t|^{\min_{i \geq 1}(\delta_i)} |Z_k|^d. \tag{4.7}$$

Suppose for any $l \neq k$, we have

$$|\frac{Z_l}{Z_k}| \geq |t|^{\sigma_1},$$

then

$$|Z_0^{\alpha_0^{(0)}} \cdots Z_n^{\alpha_n^{(0)}}| \geq |t|^{\sigma_1 d} |Z_k|^d.$$

But we choose σ_1 by

$$\sigma < \sigma_1 < \frac{1}{d} \min_{i \geq 1}(\delta_i).$$

So we get a contradiction to (4.7). Hence we get the existence part.

For the uniqueness, suppose there are $l, m \neq k$ such that

$$|\frac{Z_l}{Z_k}| < |t|^{\sigma_1}, \ \ |\frac{Z_m}{Z_k}| < |t|^{\sigma_1}$$

for t small enough, then $[Z_0, \cdots, Z_n] \in V^t_{lm}$. This is a contradiction. So we are done. ❑

Now, we will prove that for t small enough, the connected component of $M_t \setminus \cup^n_{i,j=1} V^t_{ij}$ are graphs of functions over $\tilde{P}_i$, where

$$\tilde{P}_i = P_i - \bigcup_{j \neq i} V^t_{ij}.$$

In order to do this, we first let

$$Q_i = \{[Z_0, \cdots, Z_n] \big| [Z_0, \cdots, Z_{i-1}, \underset{i}{0}, Z_{i+1}, \cdots, Z_n] \in \tilde{P}_i\},$$

for $i = 0, \cdots, n$. By the setting (1.4) and (1.5) in first section, we have

$$\psi(x_0, \cdots, x_n) = \min_{0 \leq i \leq p} (\delta + \delta_i + \alpha^{(i)}_0 x_0 + \cdots + \alpha^{(i)}_n x_n), \tag{4.8}$$

and

$$\psi_k(x) = \min_{0 \leq i \leq p} (\delta + \delta_i + \alpha^{(i)}_k x), \tag{4.9}$$

for $k = 0, \cdots, n$.

Lemma 4.3 *For $\sigma > 0$ small enough, there is a constant $\varepsilon_0 > 0$ such that the solutions of z_1 of $f = 0$ satisfies*

$$|z_1 - \varphi^k_i| \leq |\varphi^k_i||t|^{\varepsilon_0}$$

for $i = 1, \cdots, \alpha^{(i_k)}_1 - \alpha^{(i_{k+1})}_1$, $k = 0, \cdots, m-1$. Furthermore, the balls $B^k_i = \{z \in \mathbb{C} \big| |z - \varphi^k_i| \leq |\varphi^k_i||t|^{\varepsilon_0}\}$ for $i = 1, \cdots, \alpha^{(i_k)}_1 - \alpha^{(i_{k+1})}_1$, $k = 0, \cdots, m-1$, do not intersect each other if t small enough.

Proof. Without loss of generality, we assume that $(z_1, \cdots, z_n) = (\frac{Z_1}{Z_0}, \cdots, \frac{Z_n}{Z_0})$ on U_0. Then $F_t = 0$ can be written as

$$f = \sum_{i=0}^{p} a_i t^{\delta_i} z_1^{\alpha^{(i)}_1} \cdots z_n^{\alpha^{(i)}_n} = 0 \tag{4.10}$$

with $a_0 = 1$ and $\delta_0 = 0$. The sequence $(\delta_i, \alpha_1^{(i)})$, $i = 0, \cdots, p$, is assumed to be a generic sequence.

For $(z_1, \cdots, z_n) \in \tilde{P}_1 \cap U_0$, we have

$$|z_1| \geq |t|^{\sigma},$$

for $i = 2, \cdots, n$. The indices i and k are always set by $i = 1, \cdots, \alpha_1^{(i_k)} - \alpha_1^{(i_{k+1})}$, $k = 0, \cdots, m-1$, in this proof, unless otherwise stated. We choose $\varepsilon_1 > 0$ such that

$$\varepsilon_1 < \min_{0 \leq k \leq m} \min_{i \neq i_k, i_{k+1}} (\delta + \delta_i + \alpha_1^{(i)} r_{k+1} - \varphi_1(r_k + 1)).$$

Define f_k and g_k as follows

$$f_k = a_{i_k} t^{\delta_{i_k}} z_1^{\alpha_1^{(i_k)}} \cdots z_n^{\alpha_n^{(i_k)}} + a_{i_{k+1}} t^{\delta_{i_{k+1}}} z_1^{\alpha_1^{(i_{k+1})}} \cdots z_n^{\alpha_n^{(i_{k+1})}},$$

and

$$g_k = f - f_k.$$

Let φ_i^k be the $(\alpha_1^{(i_k)} - \alpha_1^{(i_{k+1})})$ - th roots of

$$-\frac{a_{i_{k+1}}}{a_{i_k}} t^{\delta_{i_{k+1}} - \delta_{i_k}} z_2^{\alpha_2^{(i_{k+1})} - \alpha_2^{(i_k)}} \cdots z_n^{\alpha_n^{(i_{k+1})} - \alpha_n^{(i_k)}}.$$

Then we have

$$|t|^{r_{k+1} + C\sigma} \leq |\varphi_i^k| \leq |t|^{r_{k+1} - C\sigma}$$

for some constant C independent of t. And we also have

$$|t|^{\delta} |g_k| \leq |t|^{\psi_1(r_{k+1}) + \varepsilon_1 - d\sigma}$$

on B_i^k and

$$|t|^{\delta} |f_k| \geq |t|^{\psi_1(r_{k+1}) + \varepsilon_0 + d\sigma}$$

on ∂B_i^k. We choose σ and ε_0 small enough such that $\varepsilon_1 - d\sigma > \frac{3}{4}\varepsilon_1$ and $\varepsilon_0 \leq \frac{1}{4}\varepsilon_1$. So we have $d\sigma < \frac{1}{4}\varepsilon_1$, and then

$$\begin{aligned} |f_k| & \geq |t|^{\psi_1(r_{k+1}) - \delta + \varepsilon_0 + d\sigma} > |t|^{\psi_1(r_{k+1}) - \delta + \varepsilon_0 + \frac{1}{4}\varepsilon_1} \\ & \geq |t|^{\psi_1(r_{k+1}) - \delta + \frac{2}{4}\varepsilon_1} > |t|^{\psi_1(r_{k+1}) - \delta + (\varepsilon_1 - d\sigma)} \geq |g_k| \end{aligned}$$

on ∂B_i^k. By the Rouché Theorem, f_k and f have the same number of solutions in B_i^k. Since f_k has only one solution in B_i^k, there is only one solution z_1 of $f = 0$ satisfies

$$|z_1 - \varphi_i^k| \leq |\varphi_i^k||t|^{\varepsilon_0}.$$

Suppose there are two balls B_i^k and $B_{i_1}^{k_1}$ such that $B_i^k \cap B_{i_1}^{k_1} \neq \phi$, then for each $z \in B_i^k \cap B_{i_1}^{k_1}$, we have

$$|\varphi_i^k - \varphi_{i_1}^{k_1}| \leq |\varphi_i^k - z| + |z - \varphi_{i_1}^{k_1}| \leq |t|^{\varepsilon_0}(|\varphi_i^k| + |\varphi_{i_1}^{k_1}|).$$

Since t small enough, we have

$$|\varphi_i^k - \varphi_{i_1}^{k_1}| < \frac{1}{2}\max\{|\varphi_i^k|, |\varphi_{i_1}^{k_1}|\}.$$

Say, $|\varphi_i^k| < |\varphi_{i_1}^{k_1}|$. This means $B_i^k \subset B_{i_1}^{k_1}$, we get a contradiction. So if t is small enough, B_i^k's do not intersect each other. ❑

Proposition 4.4 *Using the notation as above, we have*

$$\begin{aligned} &\int_{M_t \cap Q_i} \sum_{j=0}^{n} \Big(\frac{XF_t}{|\nabla F_t|^2}\Big)_j \overline{(F_t)_j}\omega^{n-1} \\ \to\ & -\delta\alpha_i^{(0)} - \int_0^\infty \psi_i'(x)(\psi_i'(x) - 1)dx, \end{aligned} \tag{4.11}$$

for $i = 0, \cdots, n$, as $t \to 0$.

Proof. In this proof, we omit the constants in an inequality for convenience. So $A \leq B$ means there is a constant C independent of t such that $A \leq CB$. It suffices to prove the case $i = 1$ because the proof of other cases are similarly. If $\alpha_1^{(0)} = 0$, then $\varphi_1' \equiv 0$, so the proposition holds automatically. Now we assume that $\alpha_1^{(0)} \geq 1$, and we only prove this property on $M_t \cap Q_1 \cap U_0$.

For the sake of simplicity, let $F = F_t$. As the setting in Lemma 4.3, the indices i, k are always running in $i = 1, \cdots, \alpha_1^{(i_k)} - \alpha_1^{(i_{k+1})}$, $k = 0, \cdots, m-1$, unless otherwise stated. For fixed i,k, attaching the B_i^k in the above lemma for each $p \in \tilde{P}_1 \cap U_0$, we get a bundle $\tilde{B}_i^k$. On each bundle $\tilde{B}_i^k$, since $|z_i| > |t|^\sigma$, we have

$$\begin{aligned}
&\sum_{j=0}^{n}\big(\frac{XF}{|\nabla F|^2}\big)_j\overline{(F)_j} = \frac{(XF)_1F_1 - (XF)F_{11}}{F_1^2} + o(1) \\
= \;& \frac{-(\delta+\delta_{i_k})\alpha_1^{(i_k)} + (\delta+\delta_{i_{k+1}})\alpha_1^{(i_{k+1})}}{\alpha_1^{(i_k)} - \alpha_1^{(i_{k+1})}} \\
&-\frac{(-\delta_{i_k}+\delta_{i_{k+1}})\big(\alpha_1^{(i_k)}(\alpha_1^{(i_k)}-1) - \alpha_1^{(i_{k+1})}(\alpha_1^{(i_{k+1})}-1)\big)}{(\alpha_1^{(i_k)} - \alpha_1^{(i_{k+1})})^2} + o(1) \\
= \;& -\delta + \frac{-\delta_{i_k}\alpha_1^{(i_k)} + \delta_{i_{k+1}}\alpha_1^{(i_{k+1})} + (\delta_{i_k} - \delta_{i_{k+1}})(\alpha_1^{(i_k)} + \alpha_1^{(i_{k+1})} - 1)}{\alpha_1^{(i_k)} - \alpha_1^{(i_{k+1})}} + o(1)
\end{aligned} \tag{4.12}$$

as $t \to 0$ for $k = 0, \cdots, m-1$, where $o(1) \to 0$ as $t \to 0$. The equation (4.12) is also true for $p \in \tilde{P}_1 \cap U_l$ for $l \neq 0$ by the same process. Hence the equation holds for $p \in \tilde{P}_1$. If $\pi : Q_1 \to \tilde{P}_1$ is the projection, and $\dfrac{\partial z_1}{\partial z_k} = -\dfrac{F_k}{F_1} \to 0$ as $t \to 0$, by (4.10), we have

$$\det \pi = o(1) \tag{4.13}$$

as $t \to 0$. Hence by (4.12), (4.13) and the main result in [3] for hypersurfaces, we have

$$\begin{aligned}
&\int_{M_t \cap Q_1} \sum_{j=0}^{n}\big(\frac{XF_t}{|\nabla F_t|^2}\big)_j\overline{(F_t)_j}\omega^{n-1} \\
= \;& \big(-\delta\alpha_1^{(0)} + \sum_{k=0}^{m-1}(\delta_{i_k} - \delta_{i_{k+1}})(\alpha_1^{(i_k)} + \alpha_1^{(i_{k+1})} - 1)\big)Vol(\mathbb{CP}^{n-1}) + o(1)
\end{aligned}$$

as $t \to 0$. We know that $Vol(\mathbb{CP}^{n-1}) = 1$. By Lemma 4.1, we get

$$\begin{aligned}
&\int_{M_t \cap Q_1} \sum_{j=0}^{n}\big(\frac{XF_t}{|\nabla F_t|^2}\big)_j\overline{(F_t)_j}\omega^{n-1} \\
\to \;& -\delta\alpha_1^{(0)} - \int_0^\infty \psi_1'(x)(\psi_1'(x) - 1)dx
\end{aligned}$$

as $t \to 0$. By the same argument, we get (4.11) holds for $i = 0, \cdots, n$. ❑

Lemma 4.5 *Let p be a fixed point in M_t and let $d(x,p)$ be the distance from $x \in \mathbb{CP}^n$ to p defined by the Fubini-Study metric. Let $B_p(\varepsilon) = \{x \in \mathbb{CP}^n \big| \, d(x,p) < \varepsilon\}$. Then there are constants C, σ independent of p and t such that*

$$\int_{M_t \cap B_p(\varepsilon)} \omega^{n-1} \le C\varepsilon^{2n-2} \log \varepsilon^{-1} \tag{4.14}$$

for t small enough, where $\varepsilon = |t|^\sigma$.

Proof. Consider the function $\rho : \mathbb{R} \to \mathbb{R}$ which is defined by

$$\rho(x) = \begin{cases} 1 & \text{, if } x \in [0,1]; \\ 0 & \text{, if } x \in \mathbb{R} \setminus [0,1]. \end{cases}$$

Then we have

$$\int_{M_t \cap B_p(\varepsilon)} \omega^{n-1} \le \int_{M_t} \rho(\frac{d(x,p)}{\varepsilon})\omega^{n-1}.$$

Since F_t be the defining function of M_t. Then in the sence of distribution, we have

$$\frac{i}{2\pi}\partial\overline{\partial} \log \frac{|F_t|^2}{(\sum_{i=0}^n |Z_i|^2)^d} = [M_t] - d\omega.$$

Then we have

$$\begin{aligned} & \int_{M_t} \rho(\frac{d(x,p)}{\varepsilon})\omega^{n-1} \\ = \quad & d\int_{\mathbb{CP}^n} \rho(\frac{d(x,p)}{\varepsilon})\omega^n + \int_{\mathbb{CP}^n} \rho(\frac{d(x,p)}{\varepsilon})\partial\overline{\partial} \log \frac{|F_t|^2}{(\sum_{i=0}^n |Z_i|^2)^d}\omega^{n-1}. \end{aligned} \tag{4.15}$$

Now we have to estimate the right hand side of (4.15). For the first term, we have

$$\int_{\mathbb{CP}^n} \rho(\frac{d(x,p)}{\varepsilon})\omega^n \le C\varepsilon^{2n}. \tag{4.16}$$

Assume that $p \in U_0 = \{[Z_0, \cdots, Z_n] \big| \, |Z_0| > \frac{1}{2}|Z_j|, \, j = 1, \cdots, n\}$. Then by (4.5), we have

$$F_t = t^\delta Z_0^d f_t,$$

where $f_t \to f_0 = z_1^{\alpha_1^{(0)}} \cdots z_n^{\alpha_n^{(0)}} \not\equiv 0$. Note that f_t is defined in Lemma 4.3. If we define $dV_0 = (\frac{i}{2\pi})^n dz_1 \wedge d\overline{z}_1 \wedge \cdots \wedge dz_n \wedge d\overline{z}_n$ is the Euclidean measure and

$|z|^2 = |z_1|^2 + \cdots + |z_n|^2$, using integration by part, we have

$$\begin{aligned} & \int_{\mathbb{CP}^n} \rho(\frac{d(x,p)}{\varepsilon}) \frac{i}{2\pi} \partial\overline{\partial} \log \frac{|F_t|^2}{(\sum_{i=0}^n |Z_i|^2)^d} \omega^{n-1} \\ \leq \quad & C\varepsilon^{2n} + \frac{C}{\varepsilon^2} \Big| \int_{|z|\leq 2\varepsilon} \log |f_t| dV_0 \Big|. \end{aligned} \tag{4.17}$$

For the second term of the right hand side of (4.17), we have

$$\frac{C}{\varepsilon^2} \Big| \int_{|z|\leq 2\varepsilon} \log |f_t| dV_0 \Big| = C\varepsilon^{2n-2} \log \varepsilon^{-1} + C\varepsilon^{2n-2} \Big| \int_{|z|\leq 2} \log |\tilde{f}_t| dV_0 \Big|, \tag{4.18}$$

where $\tilde{f}_t(z_1, \cdots, z_n) = f_t(\varepsilon z_1, \cdots, \varepsilon z_n)/\varepsilon^{\alpha_1^{(0)} + \cdots + \alpha_n^{(0)}}$. If σ is small enough, by (3.5) again, we have $\tilde{f}_t \to f_0 = z_1^{\alpha_1^{(0)}} \cdots z_n^{\alpha_n^{(0)}} \not\equiv 0$. Phong and Sturm [14] showed that

$$\int_{|z|\leq 2} \log |f_t|^{-1} dV_0 \leq C \tag{4.19}$$

for t small enough. By (4.15), (4.16), (4.17), (4.18) and (4.19), we have

$$\begin{aligned} \int_{M\cap B_p(\varepsilon)} \omega^{n-1} & \leq \int_{M_t} \rho(\frac{d(x,p)}{\varepsilon}) \omega^{n-1} \\ & = d \int_{\mathbb{CP}^n} \rho(\frac{d(x,p)}{\varepsilon}) \omega^n + \int_{\mathbb{CP}^n} \rho(\frac{d(x,p)}{\varepsilon}) \frac{i}{2\pi} \partial\overline{\partial} \log \frac{|F_t|^2}{(\sum_{i=0}^n |Z_i|^2)^d} \omega^{n-1} \\ & \leq C\varepsilon^{2n} + C\varepsilon^{2n} + \frac{C}{\varepsilon^2} \Big| \int_{|z|\leq 2\varepsilon} \log |f_t| dV_0 \Big| \\ & = C\varepsilon^{2n} + C\varepsilon^{2n} + C\varepsilon^{2n-2} \log \varepsilon^{-1} + C\varepsilon^{2n-2} \Big| \int_{|z|\leq 2} \log |\tilde{f}_t| dV_0 \Big| \\ & \leq C\varepsilon^{2n-2} \log \varepsilon^{-1}. \end{aligned}$$

Note that in this proof, $A \leq B$ means there is a constant C such that $A \leq CB$. ❑

Lemma 4.6 *There exists a constant $C > 0$ such that for t small*

$$\sum_{i\neq j} \int_{V_{ij}^t \cap M_t} \omega^{n-1} \leq C|t|^{2\sigma} \log |t|^{-1}.$$

Proof. Let $\varepsilon = |t|^\sigma$. Fix i, j, clearly, $\{B_p(\varepsilon) \big| p \in P_{ij}\}$ be an open covering of P_{ij}. There is a constant C_0 independent of ε such that we can choose $p_1, \cdots, p_m \in P_{ij}$, where $m = [\frac{C_0}{\varepsilon^{2n-4}}]$, satisfying

$$\cup_m^{k=1} B_{p_k}(\varepsilon) \supset P_{ij}.$$

By the definition of V_{ij}^t, we have

$$\cup_m^{k=1} B_{p_k}(2\varepsilon) \supset V_{ij}^t.$$

Hence we have

$$\int_{V_{ij}^t \cap M_t} \omega^{n-1} \leq \sum_{k=1}^{m} \int_{M_t \cap B_{p_k}(2\varepsilon)} \omega^{n-1}.$$

By Lemma 4.5, we have

$$\begin{aligned}\int_{V_{ij}^t \cap M_t} \omega^{n-1} &\leq \sum_{k=1}^{m} (C\varepsilon^{2n-2} \log \varepsilon^{-1}) \\ &\leq \frac{C}{\varepsilon^{2n-4}} \varepsilon^{2n-2} \log \varepsilon^{-1} \\ &= C|t|^{2\sigma} \log \varepsilon^{-1}.\end{aligned}$$

Then

$$\begin{aligned}\sum_{i \neq j} \int_{V_{ij}^t \cap M_t} \omega^{n-1} &\leq \sum_{i \neq j} C|t|^{2\sigma} \log \varepsilon^{-1} \\ &= C|t|^{2\sigma} \log |t|^{-1}.\end{aligned}$$

❑

Lemma 4.7 *There exists a constant C independent of t such that for any measurable subset E of M_t*

$$\left| \int_E \partial\varphi \wedge \overline{\partial}\theta \wedge \omega^{n-2} \right| \leq C\sqrt{\log |t|^{-1}} \cdot \sqrt{meas(E)}$$

where

$$\varphi = \log \frac{|\nabla F|^2}{(\sum_{i=0}^{n} |Z_i|^2)^{(d-1)}},$$

and

$$\theta = -\frac{\sum_{i=0}^{n} \lambda_i |Z_i|^2}{\sum_{i=0}^{n} |Z_i|^2}.$$

Proof. Since M_t is a submanifold, the Ricci curvature has an upper bound. So by (3.1), there exists a constant C such that

$$-\frac{i}{2\pi} \partial\overline{\partial}\varphi \leq C\omega. \tag{4.20}$$

By the setting in section 1, we know that $[t^{\lambda_0}Z_0, \cdots, t^{\lambda_n}Z_n] \in M_t$ if and only if $[Z_0, \cdots, Z_n] \in M$, so we have

$$|\nabla F_t|^2(t^{\lambda_0}Z_0, \cdots, t^{\lambda_n}Z_n) = \sum_{l=0}^{n} |t|^{-2\lambda_l}|F_l|^2(Z_0, \cdots, Z_n).$$

Since M is smooth, we have an estimate

$$-C\log|t|^{-1} \leq |\varphi| \leq C\log|t|^{-1}$$

for some constant C. By (4.20), using integration by parts, we have

$$\int_{M_t} |\nabla\varphi|^2\omega^{n-1} \leq C\int_{M_t}(|\varphi| + \log|t|^{-1})\omega^{n-1} \leq C\log|t|^{-1}.$$

Since E is a measurable subset of M_t, by Cauchy inequality, we have

$$\left|\int_E \partial\varphi \wedge \overline{\partial}\theta \wedge \omega^{n-2}\right| \leq \int_E |\partial\varphi| \leq C\sqrt{\log|t|^{-1}}\sqrt{meas(E)}.$$

❑

Proof of Theorem1.8. By Proposition 4.4, we have

$$\begin{aligned} &\int_{M_t\cap Q_i} \sum_{j=0}^{n} \Big(\frac{XF_t}{|\nabla F_t|^2}\Big)_j \overline{(F_t)_j}\omega^{n-1} \\ &\to \quad -\delta\alpha_i^{(0)} - \int_0^\infty \psi_i'(x)(\psi_i'(x)-1)dx, \end{aligned}$$

for $i = 0, \cdots, n$, as $t \to 0$. So

$$\begin{aligned} &\int_{M_t\cap(\cup_{i=0}^n Q_i)} \sum_{j=0}^{n} \Big(\frac{XF_t}{|\nabla F_t|^2}\Big)_j \overline{(F_t)_j}\omega^{n-1} \\ &= \quad -\delta d - \sum_{i=0}^{n}\int_0^\infty \psi_i'(x)(\psi_i'(x)-1)dx + o(1) \end{aligned} \tag{4.21}$$

as $t \to 0$. By (3.27) in Lemma 3.3, we have

$$\begin{aligned} &\int_{M_t\setminus(\cup_{i=0}^n Q_i)} \frac{i}{2\pi}\partial\varphi \wedge \overline{\partial}\theta \wedge \omega^{n-2} \\ = \quad &-\frac{1}{n-1}\int_{M_t\setminus(\cup_{i=0}^n Q_i)} \Big(\sum_{j=0}^{n}\Big(\frac{XF_t}{|\nabla F_t|^2}\Big)_j \overline{(F_t)_j} - \frac{\sum_{i=0}^n \lambda_i|(F_t)_i|^2}{|\nabla F_t|^2} + (d-1)\theta\Big)\omega^{n-1}. \end{aligned}$$

Note that θ and $\dfrac{\sum_{i=0}^{n}\lambda_i|(F_t)_i|^2}{|\nabla F_t|^2}$ are bounded, we have

$$\int_{M_t\setminus(\cup_{i=0}^{n}Q_i)}|\sum_{j=0}^{n}\big(\frac{XF_t}{|\nabla F_t|^2}\big)_j\overline{(F_t)_j}|\omega^{n-1}\leq\int_{M_t\setminus(\cup_{i=0}^{n}Q_i)}(|\partial\varphi|+1)\omega^{n-1}.$$

By Lemma 4.7, we have

$$\begin{aligned}&\int_{M_t\setminus(\cup_{i=0}^{n}Q_i)}(|\partial\varphi|+1)\omega^{n-1}\\ \leq\;& C\sqrt{\log|t|^{-1}}\sqrt{meas(M_t\setminus\cup_{i=0}^{n}Q_i)}+meas(M_t\setminus\cup_{i=0}^{n}Q_i).\end{aligned}$$

Consider $[Z_0,\cdots,Z_n]\in M_t\setminus\cup_{i=0}^{n}Q_i$, without loss of generality, we may assume that $[Z_0,\cdots,Z_n]\in U_0$. By (4.7) in Lemma 4.2, we can find $k\neq 0$ such that

$$|Z_k|\leq|t|^{\sigma}|Z_0|$$

for t small enough. Since $[Z_0,\cdots,Z_n]\notin Q_k$, there exists a $j\neq 0,k$ such that

$$|Z_j|\leq|t|^{\sigma}|Z_0|.$$

So we get $[Z_0,\cdots,Z_n]\in V_{jk}^{Ct}$ for some constant C. Hence

$$M_t\setminus\cup_{i=0}^{n}Q_i\subset\cup_{i\neq j}V_{ij}^{Ct}. \tag{4.22}$$

By Lemma 4.6, we have

$$\begin{aligned}&\int_{M_t\setminus(\cup_{i=0}^{n}Q_i)}\sum_{A=0}^{n}\big(\frac{XF_t}{|\nabla F_t|^2}\big)_A\overline{(F_t)_A}\omega^{n-1}\\ \leq\;&\sum_{i\neq j}\int_{M_t\cap V_{ij}^{Ct}}\sum_{A=0}^{n}\big(\frac{XF_t}{|\nabla F_t|^2}\big)_A\overline{(F_t)_A}\omega^{n-1}=o(1)\end{aligned} \tag{4.23}$$

as $t\to 0$. By (4.21) and (4.23), we have

$$\begin{aligned}&\int_{M_t}\sum_{j=0}^{n}\big(\frac{XF_t}{|\nabla F_t|^2}\big)_j\overline{(F_t)_j}\omega^{n-1}\\ =\;&\int_{M_t\cap(\cup_{i=0}^{n}Q_i)}\sum_{j=0}^{n}\big(\frac{XF_t}{|\nabla F_t|^2}\big)_j\overline{(F_t)_j}\omega^{n-1}+\int_{M_t\setminus(\cup_{i=0}^{n}Q_i)}\sum_{j=0}^{n}\big(\frac{XF_t}{|\nabla F_t|^2}\big)_j\overline{(F_t)_j}\omega^{n-1}\\ =\;&-\delta d-\sum_{i=0}^{n}\int_0^{\infty}\psi_i'(x)(\psi_i'(x)-1)dx+o(1)\end{aligned}$$

as $t \to 0$. If M_0 is defined as the zero set of $Z_0^{\alpha_0^{(0)}} \cdots Z_n^{\alpha_n^{(0)}} = 0$ counting the multiplicity, since θ is a bounded function, we have

$$\int_{M_t} \theta\omega^{n-1} = \int_{M_0} \delta\omega^{n-1} + o(1)$$

as $t \to 0$. Zhiqin Lu [10, Theorem 5.1] showed that

$$\int_{M_0} \theta\omega^{n-1} = -\frac{\delta}{n}.$$

By (3.31) in Theorem 3.4, we have

$$t\frac{d}{dt}\mathcal{M}(t) = \frac{2}{d}\big(\frac{\delta(n+1)(d-1)}{n} + \sum_{i=0}^{n}\int_0^\infty \psi_i'(x)(\psi_i'(x)-1)dx\big) + o(1)$$

as $t \to 0$. Hence

$$\begin{aligned} &\lim_{t\to 0} t\frac{d}{dt}\mathcal{M}(t) \\ = \ & \frac{2}{d}\big(-\frac{\lambda(n+1)(d-1)}{n} + \sum_{i=0}^{n}\int_0^\infty \psi_i'(x)(\psi_i'(x)-1)dx\big) \end{aligned}$$

for generic $(\lambda_0, \cdots, \lambda_n)$. ❑

Since for a Kähler–Einstein manifold, the K energy has a lower bound, Lu[11] give a general result of theorem 1.8.

Theorem 4.8 *(Lu) If M is a Kähler–Einstein hypersurface with positive first Chern class, then we have*

$$-\frac{\lambda(d-1)(n+1)}{n} + \sum_{i=0}^{n}\int_0^\infty \psi_i'(x)(\psi_i'(x)-1)dx \le 0$$

for any $\lambda_0, \cdots, \lambda_n \in \mathbb{R}$ with $\sum_{i=0}^{n} \lambda_i = 0$.

Here, we give two effective ways to verify the K stability for hypersurface.

Theorem 4.9 *Let M be a compact Fano hypersurface defined by the zeros of the polynomial F of degree $d \geq 1$. If one can find a sequence $(\lambda_0, \cdots, \lambda_n)$ with $\sum_{k=0}^{n} \lambda_k = 0$ such that $\lambda < 0$, then there is no Kähler–Einstein metric on M.*

Proof. By the equation (1.2), if $\lambda = \max_{0 \leq i \leq p} (\sum_{k=0}^{n} \lambda_k \alpha_k^{(i)}) < 0$, then we have $\sum_{k=0}^{n} \lambda_k \alpha_k^{(i)} < 0$, for all $0 \leq i \leq p$. By the equation (1.3) and (1.4), we have either

$$\psi_i(x) = -\sum_{k=0}^{n} \lambda_k \alpha_k^{(j)}$$

for some $0 \leq j \leq p$ or

$$\psi_i(x) = \begin{cases} -\sum_{k=0}^{n} \lambda_k \alpha_k^{(i_1)} + \alpha_k^{(i_1)} x & , 0 \leq x < b, \\ -\sum_{k=0}^{n} \lambda_k \alpha_k^{(i_2)} & , x \geq b, \end{cases}$$

for some $i_1 \neq i_2$, $0 \leq i_1, i_2 \leq p$, $x \in [0, b)$, $0 < b < \infty$. So either $\psi_i'(x) = 0$ on $[0, \infty)$ or

$$\psi_i'(x) = \begin{cases} \alpha_k^{(j)} & , 0 \leq x < b, \\ 0 & , x \geq b, \end{cases}$$

for some $0 \leq j \leq p$, $x \in [0, b)$, $0 < b < \infty$. By theorem 1.8 and the fact $\alpha_k^{(j)} \geq 1$, we hane

$$\frac{2}{d}\Big(-\frac{\lambda(d-1)(n+1)}{n} + \sum_{i=0}^{n} \int_0^{\infty} \psi_i'(x)(\psi_i'(x) - 1) dx \Big) > 0.$$

Hence M is not K stable. By theorem 4.8, we know that there is no Kähler–Einstein metric on M. ❑

Theorem 4.10 *Let M be a compact Fano hypersurface on $\mathbb{CP}^n$ defined by the zeros of polynomial $F(Z_0, \cdots, Z_n)$ of degree $d \geq 1$. Suppose that for some $k = 0, \cdots, n$, we have $\alpha_k^{(i)} = 0$ for all $i = 0, \cdots, p$. Then there is no Kähler–Einstein metric on M.*

Proof. Without losing generality, we assume that F miss the term Z_0. Write

$$F(Z_0, \cdots, Z_n) = \sum_{i=0}^{p} a_i Z_1^{\alpha_1^{(i)}} \cdots Z_n^{\alpha_n^{(i)}}.$$

Take $\lambda_i = -i$, $i = 1, \cdots, n$, and $\lambda_0 = \dfrac{n(n+1)}{2}$. From the equation (1.2), we have

$$\lambda = \max_{0 \le i \le p} \Big(\sum_{k=0}^{n} \lambda_k \alpha_k^{(i)}\Big) = \max_{0 \le i \le p} \Big(\sum_{k=1}^{n} \lambda_k \alpha_k^{(i)}\Big) < 0.$$

By theorem 4.9, there is no Kähler–Einstein metric on M. ❑

5 Some Examples

***Example* 5.1**. In $\mathbb{CP}^2$, let M be defined by the zeros of the polynomial

$$F(z_0, z_1, z_2) = z_0^2 + z_1^2 + z_2^2 + 2z_0z_1 + 2z_0z_2 + 2z_1z_2$$

of degree 2. Let $\lambda_0, \lambda_1, \lambda_2$ be 3 rational numbers sum to 0. By the equation (1.2), we have

$$\lambda = \max\{2\lambda_0, 2\lambda_1, 2\lambda_2, \lambda_0 + \lambda_1, \lambda_0 + \lambda_2, \lambda_1 + \lambda_2\} > 0.$$

And by (1.3), (1.4), we have

$$\begin{aligned} \psi(x_0, x_1, x_2) = \quad & \min\{-2\lambda_0 + 2x_0, -2\lambda_1 + 2x_1, -2\lambda_2 + 2x_2, \\ & -\lambda_0 - \lambda_1 + x_0 + x_1, -\lambda_0 - \lambda_2 + x_0 + x_2, -\lambda_1 - \lambda_2 + x_1 + x_2\}. \end{aligned}$$

$$\psi_0(x) = \min\{-2\lambda_0 + 2x, -2\lambda_1, -2\lambda_2, -\lambda_0 - \lambda_1 + x, -\lambda_0 - \lambda_2 + x, -\lambda_1 - \lambda_2\}.$$

$$\psi_1(x) = \min\{-2\lambda_0, -2\lambda_1 + 2x, -2\lambda_2, -\lambda_0 - \lambda_1 + x, -\lambda_0 - \lambda_2, -\lambda_1 - \lambda_2 + x\}.$$

$$\psi_2(x) = \min\{-2\lambda_0, -2\lambda_1, -2\lambda_2 + 2x, -\lambda_0 - \lambda_1, -\lambda_0 - \lambda_2 + x, -\lambda_1 - \lambda_2 + x\}.$$

For the 3 numbers λ_0, λ_1 and λ_2, we must consider 3 cases:
Case1: $\lambda_0 = 0$, $\lambda_1 > 0$, $\lambda_2 < 0$.

In this case, since $2\lambda_0 = 0 = \lambda_1 + \lambda_2$, $(\lambda_0, \lambda_1, \lambda_2)$ not be generic. But theorem 4.8 shows that (1.5) is valid for any choice of $\lambda_0, \cdots, \lambda_n \in \mathbb{R}$.

$$\lambda = \max\{0, 2\lambda_1, -2\lambda_1, \lambda_1, -\lambda_1, 0\} = 2\lambda_1 > 0.$$

$$\psi_0(x) = \min\{2x, -2\lambda_1, 2\lambda_1, -\lambda_1 + x, \lambda_1 + x, 0\} = -2\lambda_1 \text{ as } x \geq 0,$$

$$\begin{aligned} \psi_1(x) \quad &= \min\{0, -2\lambda_1 + 2x, 2\lambda_1, -\lambda_1 + x, \lambda_1, x\} \\ &= \begin{cases} -2\lambda_1 + 2x & \text{, if } 0 \leq x < \lambda_1, \\ 0 & \text{, if } x \geq \lambda_1, \end{cases} \end{aligned}$$

$$\psi_2(x) = \min\{0, -2\lambda_1, 2\lambda_1 + 2x, -\lambda_1, \lambda_1 + x, x\} = -2\lambda_1 \text{ as } x \geq 0.$$

So $\psi_0'(x) = 0$ as $x \geq 0$, $\psi_2'(x) = 0$ as $x \geq 0$, and

$$\psi_1'(x) = \begin{cases} 2 & \text{, if } \ 0 \leq x < \lambda_1, \\ 0 & \text{, if } \ x \geq \lambda_1. \end{cases}$$

By the equation (1.5), we have

$$\begin{aligned} & \lim_{t\to 0} t\frac{d}{dt}\mathcal{M}(t) \\ = \ & \frac{2}{2}(-\frac{2\lambda_1 \cdot 1 \cdot 3}{2} + \int_0^{\lambda_1} 2 \cdot 1 dx + \int_{\lambda_1}^{\infty} 0 \cdot (-1) dx + \int_0^{\infty} 0 \cdot (-1) dx + \int_0^{\infty} 0 \cdot (-1) dx) \\ = \ & -3\lambda_1 + 2\lambda_1 \ = -\lambda_1 \ < 0. \end{aligned}$$

Case2: $\lambda_0 > 0$, $\lambda_1 > 0$, $\lambda_2 < 0$.

We may assume that $\lambda_0 > \lambda_1$ and $\lambda_2 = -\lambda_0 - \lambda_1$.

$$\lambda = \max\{2\lambda_0, 2\lambda_1, -2\lambda_0 - 2\lambda_1, \lambda_0 + \lambda_1, -\lambda_1, -\lambda_0\} = 2\lambda_0 > 0.$$

$$\begin{aligned} \psi_0(x) &= \min\{-2\lambda_0 + 2x, -2\lambda_1, 2\lambda_0 + 2\lambda_1, -\lambda_0 - \lambda_1 + x, \lambda_1 + x, \lambda_0\} \\ &= \begin{cases} -2\lambda_0 + 2x & \text{, if } \ 0 \leq x < \lambda_0 - \lambda_1, \\ -2\lambda_1 & \text{, if } \ x \geq \lambda_0 - \lambda_1. \end{cases} \\ \psi_1(x) &= \min\{-2\lambda_0, -2\lambda_1 + 2x, 2\lambda_0 + 2\lambda_1, -\lambda_0 - \lambda_1 + x, \lambda_1, \lambda_0 + x\} \\ &= -2\lambda_0 \text{ as } x \geq 0. \end{aligned}$$

$$\begin{aligned} \psi_2(x) &= \min\{-2\lambda_0, -2\lambda_1, 2\lambda_0 + 2\lambda_1 + 2x, -\lambda_0 - \lambda_1, \lambda_1 + x, \lambda_0 + x\} \\ &= -2\lambda_0 \text{ as } x \geq 0. \end{aligned}$$

So $\psi_1'(x) = 0$ as $x \geq 0$, $\psi_2'(x) = 0$ as $x \geq 0$, and

$$\psi_0'(x) = \begin{cases} 2 & \text{, if } \ 0 \leq x < \lambda_0 - \lambda_1, \\ 0 & \text{, if } \ x \geq \lambda_0 - \lambda_1. \end{cases}$$

By theorem 1.8, we have

$$\begin{aligned}
&\lim_{t\to 0} t\frac{d}{dt}\mathcal{M}(t)\\
= \quad & \frac{2}{2}(-\frac{2\lambda_0\cdot 1\cdot 3}{2} + \int_0^{\lambda_0-\lambda_1} 2\cdot 1dx + \int_{\lambda_0-\lambda_1}^{\infty} 0\cdot(-1)dx + \int_0^{\infty} 0\cdot(-1)dx + \int_0^{\infty} 0\cdot(-1)dx)\\
= \quad & -3\lambda_0 + 2\lambda_0 - 2\lambda_1 \ = -\lambda_0 - 2\lambda_1 \ < 0.
\end{aligned}$$

Case3: $\lambda_0 < 0,\ \lambda_1 < 0,\ \lambda_2 > 0.$

We may assume that $\lambda_0 > \lambda_1$ and $\lambda_2 = -\lambda_0 - \lambda_1$.

$$\lambda = \max\{2\lambda_0, 2\lambda_1, -2\lambda_0 - 2\lambda_1, \lambda_0+\lambda_1, -\lambda_1, -\lambda_0\} = -2\lambda_0 - 2\lambda_1 > 0.$$

$$\begin{aligned}
\psi_0(x) &= \min\{-2\lambda_0+2x, -2\lambda_1, 2\lambda_0+2\lambda_1, -\lambda_0-\lambda_1+x, \lambda_1+x, \lambda_0\}\\
&= 2\lambda_0+2\lambda_1 \text{ if } x\geq 0.\\
\psi_1(x) &= \min\{-2\lambda_0, -2\lambda_1+2x, 2\lambda_0+2\lambda_1, -\lambda_0-\lambda_1+x, \lambda_1, \lambda_0+x\}\\
&= 2\lambda_0+2\lambda_1 \text{ if } x\geq 0.\\
\psi_2(x) &= \min\{-2\lambda_0, -2\lambda_1, 2\lambda_0+2\lambda_1+2x, -\lambda_0-\lambda_1, \lambda_1+x, \lambda_0+x\}\\
&= \begin{cases} 2\lambda_0+2\lambda_1+2x & ,\text{if } \ 0\leq x < -2\lambda_0-\lambda_1,\\ -2\lambda_0 & ,\text{if } \ x \geq -2\lambda_0-\lambda_1.\end{cases}
\end{aligned}$$

So $\psi_0'(x) = 0$ as $x \geq 0$, $\psi_1'(x) = 0$ as $x \geq 0$, and

$$\psi_2'(x) = \begin{cases} 2 & ,\text{if } \ 0\leq x < -2\lambda_0-\lambda_1,\\ 0 & ,\text{if } \ x \geq -2\lambda_0-\lambda_1.\end{cases}$$

By theorem 1.8, we have

$$\begin{aligned}
&\lim_{t\to 0} t\frac{d}{dt}\mathcal{M}(t)\\
= \quad & \frac{2}{2}\Big(-\frac{(-2\lambda_0-2\lambda_1)\cdot 1\cdot 3}{2} + \int_0^{-2\lambda_0-\lambda_1} 2\cdot 1dx\Big)\\
= \quad & 3\lambda_0 + 3\lambda_1 - 4\lambda_0 - 2\lambda_1 \ = \lambda_1 - \lambda_0 \ < 0.
\end{aligned}$$

For example, let $(\lambda_0, \lambda_1, \lambda_2) = (-\frac{1}{3}, -\frac{2}{3}, 1)$. Then we have

$$\begin{aligned}
\lambda &= \max\{(-\frac{1}{3})\cdot 2, (-\frac{2}{3})\cdot 2, 1\cdot 2, (-\frac{1}{3})\cdot 1 + (-\frac{2}{3})\cdot 1, (-\frac{1}{3})\cdot 1 + 1\cdot 1, (-\frac{2}{3})\cdot 1 + 1\cdot 1\}\\
&= \max\{-\frac{2}{3}, -\frac{4}{3}, 2, -1, \frac{2}{3}, \frac{1}{3}\} = 2.
\end{aligned}$$

And

$$\psi_0(x) = \min\{\frac{2}{3}+2x, \frac{4}{3}, -2, 1+x, -\frac{2}{3}+x, -\frac{1}{3}\} = -2 \text{ as } x \geq 0,$$
$$\psi_1(x) = \min\{\frac{2}{3}, \frac{4}{3}+2x, -2, 1+x, -\frac{2}{3}, -\frac{1}{3}+x\} = -2 \text{ as } x \geq 0,$$

$$\begin{aligned}\psi_2(x) &= \min\{\frac{2}{3}, \frac{4}{3}, -2+2x, 1, -\frac{2}{3}+x, -\frac{1}{3}+x\} \\ &= \min\{\frac{2}{3}, -2+2x, -\frac{2}{3}+x\} \\ &= \begin{cases} -2+2x & \text{, if } 0 \leq x < \frac{4}{3}, \\ \frac{2}{3} & \text{, if } x \geq \frac{4}{3}. \end{cases}\end{aligned}$$

So $\psi_0'(x) = 0$ as $x \geq 0$, $\psi_1'(x) = 0$ as $x \geq 0$, and

$$\psi_2'(x) = \begin{cases} 2 & \text{, if } 0 \leq x < \frac{4}{3}, \\ 0 & \text{, if } x \geq \frac{4}{3}. \end{cases}$$

By theorem 1.8, we have

$$\begin{aligned}&\lim_{t\to 0} t\frac{d}{dt}\mathcal{M}(t) \\ = \ & \frac{2}{2}(-\frac{2\cdot 1\cdot 3}{2} + \int_0^{\frac{4}{3}} 2\cdot 1 dx) = -3 + \frac{8}{3} \ = -\frac{1}{3} \ < 0.\end{aligned}$$

So we know that M is K stable. ❑

***Example* 5.2**. In $\mathbb{CP}^3$, let M be defined by the zeros of the polynomial

$$F(z_0, z_1, z_2, z_3) = z_2^3 + 5z_1z_2z_3 - 7z_0^2z_3 + 2z_2^2z_3$$

of degree 3. Let $(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = (\frac{1}{5}, \frac{4}{5}, -\frac{2}{5}, -\frac{3}{5})$. Then we have

$$\lambda = \max\{-\frac{6}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{7}{5}\} = -\frac{1}{5} < 0.$$

$$\psi(x_0, x_1, x_2, x_3) = \min\{\frac{6}{5}+3x_2, \frac{1}{5}+x_1+x_2+x_3, \frac{1}{5}+2x_0+x_3, \frac{7}{5}+2x_2+x_3\}.$$

And

$$\psi_0(x) = \min\{\frac{6}{5}, \frac{1}{5}, \frac{1}{5}+2x, \frac{7}{5}\} = \frac{1}{5} \text{ as } x \geq 0,$$

$$\psi_1(x) = \min\{\frac{6}{5}, \frac{1}{5}+x, \frac{1}{5}, \frac{7}{5}\} = \frac{1}{5} \text{ as } x \geq 0,$$

$$\psi_2(x) = \min\{\frac{6}{5}+3x, \frac{1}{5}+x, \frac{1}{5}, \frac{7}{5}+2x\} = \frac{1}{5} \text{ as } x \geq 0,$$

$$\begin{aligned} \psi_3(x) &= \min\{\frac{6}{5}, \frac{1}{5}+x, \frac{1}{5}+x, \frac{7}{5}+x\} \\ &= \begin{cases} \frac{1}{5}+x & \text{, if } 0 \leq x < 1, \\ \frac{6}{5} & \text{, if } x \geq 1. \end{cases} \end{aligned}$$

So $\psi_0'(x) = 0$, $\psi_1'(x) = 0$, $\psi_2'(x) = 0$ as $x \geq 0$, and

$$\psi_3'(x) = \begin{cases} 1 & \text{, if } 0 \leq x < 1, \\ 0 & \text{, if } x \geq 1. \end{cases}$$

By theorem 1.8, we have

$$\begin{aligned} & \lim_{t\to 0} t\frac{d}{dt}\mathcal{M}(t) \\ = \; & \frac{2}{3}(-\frac{-\frac{1}{5}\cdot 2\cdot 4}{3} + \int_0^1 1\cdot 0 dx) = \frac{16}{45} > 0. \end{aligned}$$

So we know that M is not K stable. Since $\lambda < 0$, we can also use theorem 4.9 to get the same result quickly. ❑

***Example* 5.3**. In $\mathbb{CP}^2$, let M be defined by the zeros of the polynomial

$$F(z_0, z_1, z_2) = z_1^2 - 5z_2^2 - 3z_1z_2$$

of degree 2, F miss the term z_0. Let $(\lambda_0, \lambda_1, \lambda_2) = (\frac{1}{2}, 0, -\frac{1}{2})$. We have

$$\lambda = \max\{0, -1, -\frac{1}{2}\} = 0.$$

$$\psi(x_0, x_1, x_2) = \min\{2x_1, 1+2x_2, \frac{1}{2}+x_1+x_2\}$$

And

$$\psi_0(x) = \min\{0, 1, \frac{1}{2}\} = 0 \text{ as } x \geq 0,$$

$$\begin{aligned} \psi_1(x) &= \min\{2x, 1, \frac{1}{2}+x\} \\ &= \begin{cases} 2x & \text{, if } 0 \le x < \frac{1}{2}, \\ 1 & \text{, if } x \ge \frac{1}{2}. \end{cases} \end{aligned}$$

$$\psi_2(x) = \min\{0, 1+2x, \frac{1}{2}+x\} = 0 \text{ as } x \ge 0.$$

So $\psi_0'(x) = 0$, $\psi_2'(x) = 0$ and

$$\psi_1'(x) = \begin{cases} 2 & \text{, if } 0 \le x < \frac{1}{2}, \\ 0 & \text{, if } x \ge \frac{1}{2}. \end{cases}$$

By theorem 1.8,

$$\lim_{t\to 0} t\frac{d}{dt}\mathcal{M}(t) = \frac{2}{2}\left(-\frac{0\cdot 1\cdot 3}{2} + \int_0^{\frac{1}{2}} 2dx\right) = 1 > 0.$$

Hence there is no Kähler–Einstein metric on M. This example is satisfies the conclusion of theorem 4.10. ❑

***Example* 5.4**. In $\mathbb{CP}^3$, let M be defined by the zeros of the polynomial

$$F(z_0, z_1, z_2, z_3) = z_0^3 + 8z_1^2 z_2 - 6z_0^2 z_1 + 5z_2^2 z_3$$

of degree 3. Let $(\lambda_0, \lambda_1, \lambda_2, \lambda_3) = (-10, -17, -20, 47)$. Then we have

$$\lambda = \max\{-30, -54, -37, 7\} = 7 > 0.$$

$$\psi(x_0, x_1, x_2, x_3) = \min\{30 + 3x_0, 54 + 2x_1 + x_2, 37 + 2x_0 + x_1, -7 + 2x_2 + x_3\}.$$

And

$$\psi_0(x) = \min\{30 + 3x, 54, 37 + 2x, -7\} = -7 \text{ as } x \ge 0,$$

$$\psi_1(x) = \min\{30, 54 + 2x, 37 + x, -7\} = -7 \text{ as } x \ge 0,$$

$$\begin{aligned} \psi_2(x) &= \min\{30, 54 + x, 37, -7 + 2x\} \\ &= \begin{cases} -7 + 2x & \text{, if } 0 \le x < \frac{37}{2}, \\ 30 & \text{, if } x \ge \frac{37}{2}. \end{cases} \end{aligned}$$

$$\begin{aligned}\psi_3(x) &= \min\{30, 54, 37, -7+x\} \\ &= \begin{cases} -7+x & ,\text{if } \ 0 \le x < 37, \\ 30 & ,\text{if } \ x \ge 37. \end{cases}\end{aligned}$$

So $\psi_0'(x) = 0$, $\psi_1'(x) = 0$ as $x \ge 0$, and

$$\psi_2'(x) = \begin{cases} 2 & ,\text{if } \ 0 \le x < \frac{37}{2}, \\ 0 & ,\text{if } \ x \ge \frac{37}{2}, \end{cases}$$

$$\psi_3'(x) = \begin{cases} 1 & ,\text{if } \ 0 \le x < 37, \\ 0 & ,\text{if } \ x \ge 37. \end{cases}$$

By theorem 1.8, we have

$$\begin{aligned} & \lim_{t\to 0} t\frac{d}{dt}\mathcal{M}(t) \\ = \ & \frac{2}{3}\left(-\frac{7\cdot 2\cdot 4}{3} + \int_0^{\frac{37}{2}} 2\cdot 1 dx + \int_0^{37} 1\cdot 0 dx\right) = \frac{110}{9} > 0. \end{aligned}$$

By theorem 4.8, there is no Kähler–Einstein metric on M. ❑

***Example* 5.5**. In $\mathbb{CP}^{100}$, let M be defined by the zeros of the polynomial

$$\begin{aligned} F = \ & 3Z_0^{18}Z_1^{32} - 7Z_2^{36}Z_{84}^{14} + 98Z_3^{10}Z_{22}^{22}Z_{71}^{18} + 78Z_9^{2}Z_{33}^{4}Z_{68}^{44} + 101Z_{16}^{8}Z_{23}^{3}Z_{78}^{39} \\ & -98Z_{22}^{31}Z_{37}^{19} + 74Z_{29}^{6}Z_{30}^{6}Z_{79}^{38} + 69Z_{29}^{26}Z_{33}^{17}Z_{70}^{7} + 36Z_{37}^{19}Z_{71}^{18}Z_{99}^{13} + 61Z_{60}^{18}Z_{79}^{32} \end{aligned}$$

of degree 50. Let $\lambda_0 = 0$, $\lambda_i = \frac{2}{i+1}, i = 1, 3, \cdots, 99$, $\lambda_i = -\frac{2}{i}, i = 2, 4, \cdots, 100$. Then we have

$$\lambda = \max\{32, -\frac{109}{3}, \frac{7}{2}, -\frac{56}{85}, -\frac{7}{4}, -\frac{20}{11}, \frac{19}{20}, \frac{5}{3}, \frac{25}{44}, \frac{1}{5}\} = 32.$$

$$\begin{aligned} \psi(x_0, \cdots, x_{100}) = \ & \min\{-32 + 18x_0 + 32x_1, \frac{109}{3} + 36x_2 + 14x_{84}, \\ & -\frac{7}{2} + 10x_3 + 22x_{22} + 18x_{71}, \frac{56}{85} + 2x_9 + 4x_{33} + 44x_{68}, \\ & \frac{7}{4} + 8x_{16} + 3x_{23} + 39x_{78}, \frac{20}{11} + 31x_{22} + 19x_{37}, \\ & -\frac{19}{20} + 6x_{29} + 6x_{30} + 38x_{79}, -\frac{5}{3} + 26x_{29} + 17x_{33} + 7x_{70}, \\ & -\frac{25}{44} + 19x_{37} + 18x_{71} + 13x_{99}, -\frac{1}{5} + 18x_{60} + 32x_{79}\}. \end{aligned}$$

Now, we have to calculate $\psi_i(x), i = 0, \cdots, 100$.

$$
\begin{aligned}
\psi_0(x) &= \min\{-32+18x, \frac{109}{3}, -\frac{7}{2}, \frac{56}{85}, \frac{7}{4}, \frac{20}{11}, -\frac{19}{20}, -\frac{5}{3}, -\frac{25}{44}, -\frac{1}{5}\} \\
&= \begin{cases} -32+18x & ,\text{if } \ 0 \le x < \frac{19}{12}, \\ -\frac{7}{2} & ,\text{if } \ x \ge \frac{19}{12}, \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\psi_1(x) &= \min\{-32+32x, \frac{109}{3}, -\frac{7}{2}, \frac{56}{85}, \frac{7}{4}, \frac{20}{11}, -\frac{19}{20}, -\frac{5}{3}, -\frac{25}{44}, -\frac{1}{5}\} \\
&= \begin{cases} -32+32x & ,\text{if } \ 0 \le x < \frac{57}{64}, \\ -\frac{7}{2} & ,\text{if } \ x \ge \frac{57}{64}, \end{cases}
\end{aligned}
$$

$$\psi_2(x) = \min\{-32, \frac{109}{3}+36x, -\frac{7}{2}, \frac{56}{85}, \frac{7}{4}, \frac{20}{11}, -\frac{19}{20}, -\frac{5}{3}, -\frac{25}{44}, -\frac{1}{5}\} = -32 \text{ as } x \ge 0,$$

$$\psi_3(x) = \min\{-32, \frac{109}{3}, -\frac{7}{2}+10x, \frac{56}{85}, \frac{7}{4}, \frac{20}{11}, -\frac{19}{20}, -\frac{5}{3}, -\frac{25}{44}, -\frac{1}{5}\} = -32 \text{ as } x \ge 0,$$

$$\psi_9(x) = \min\{-32, \frac{109}{3}, -\frac{7}{2}, \frac{56}{85}+2x, \frac{7}{4}, \frac{20}{11}, -\frac{19}{20}, -\frac{5}{3}, -\frac{25}{44}, -\frac{1}{5}\} = -32 \text{ as } x \ge 0,$$

$$\psi_{16}(x) = \min\{-32, \frac{109}{3}, -\frac{7}{2}, \frac{56}{85}, \frac{7}{4}+8x, \frac{20}{11}, -\frac{19}{20}, -\frac{5}{3}, -\frac{25}{44}, -\frac{1}{5}\} = -32 \text{ as } x \ge 0,$$

$$\psi_{22}(x) = \min\{-32, \frac{109}{3}, -\frac{7}{2}+22x, \frac{56}{85}, \frac{7}{4}, \frac{20}{11}+31x, -\frac{19}{20}, -\frac{5}{3}, -\frac{25}{44}, -\frac{1}{5}\} = -32 \text{ as } x \ge 0,$$

$$\psi_{23}(x) = \min\{-32, \frac{109}{3}, -\frac{7}{2}, \frac{56}{85}, \frac{7}{4}+3x, \frac{20}{11}, -\frac{19}{20}, -\frac{5}{3}, -\frac{25}{44}, -\frac{1}{5}\} = -32 \text{ as } x \ge 0,$$

$$\psi_{29}(x) = \min\{-32, \frac{109}{3}, -\frac{7}{2}, \frac{56}{85}, \frac{7}{4}, \frac{20}{11}, -\frac{19}{20}+6x, -\frac{5}{3}+26x, -\frac{25}{44}, -\frac{1}{5}\} = -32 \text{ as } x \ge 0,$$

$$\psi_{30}(x) = \min\{-32, \frac{109}{3}, -\frac{7}{2}, \frac{56}{85}, \frac{7}{4}, \frac{20}{11}, -\frac{19}{20}+6x, -\frac{5}{3}, -\frac{25}{44}, -\frac{1}{5}\} = -32 \text{ as } x \ge 0,$$

$$\psi_{33}(x) = \min\{-32, \frac{109}{3}, -\frac{7}{2}, \frac{56}{85}+4x, \frac{7}{4}, \frac{20}{11}, -\frac{19}{20}, -\frac{5}{3}+17x, -\frac{25}{44}, -\frac{1}{5}\} = -32 \text{ as } x \ge 0,$$

$$\psi_{37}(x) = \min\{-32, \frac{109}{3}, -\frac{7}{2}, \frac{56}{85}, \frac{7}{4}, \frac{20}{11}+19x, -\frac{19}{20}, -\frac{5}{3}, -\frac{25}{44}+19x, -\frac{1}{5}\} = -32 \text{ as } x \ge 0,$$

$$\psi_{60}(x) = \min\{-32, \frac{109}{3}, -\frac{7}{2}, \frac{56}{85}, \frac{7}{4}, \frac{20}{11}, -\frac{19}{20}, -\frac{5}{3}, -\frac{25}{44}, -\frac{1}{5}+18x\} = -32 \text{ as } x \ge 0,$$

$$\psi_{68}(x) = \min\{-32, \frac{109}{3}, -\frac{7}{2}, \frac{56}{85}+44x, \frac{7}{4}, \frac{20}{11}, -\frac{19}{20}, -\frac{5}{3}, -\frac{25}{44}, -\frac{1}{5}\} = -32 \text{ as } x \ge 0,$$

$$\psi_{70}(x) = \min\{-32, \frac{109}{3}, -\frac{7}{2}, \frac{56}{85}, \frac{7}{4}, \frac{20}{11}, -\frac{19}{20}, -\frac{5}{3}+7x, -\frac{25}{44}, -\frac{1}{5}\} = -32 \text{ as } x \ge 0,$$

$$\psi_{71}(x) = \min\{-32, \frac{109}{3}, -\frac{7}{2}+18x, \frac{56}{85}, \frac{7}{4}, \frac{20}{11}, -\frac{19}{20}, -\frac{5}{3}, -\frac{25}{44}+18x, -\frac{1}{5}\} = -32 \text{ as } x \geq 0,$$

$$\psi_{78}(x) = \min\{-32, \frac{109}{3}, -\frac{7}{2}, \frac{56}{85}, \frac{7}{4}+39x, \frac{20}{11}, -\frac{19}{20}, -\frac{5}{3}, -\frac{25}{44}, -\frac{1}{5}\} = -32 \text{ as } x \geq 0,$$

$$\psi_{79}(x) = \min\{-32, \frac{109}{3}, -\frac{7}{2}, \frac{56}{85}, \frac{7}{4}, \frac{20}{11}, -\frac{19}{20}+38x, -\frac{5}{3}, -\frac{25}{44}, -\frac{1}{5}+32x\} = -32 \text{ as } x \geq 0,$$

$$\psi_{84}(x) = \min\{-32, \frac{109}{3}+14x, -\frac{7}{2}, \frac{56}{85}, \frac{7}{4}, \frac{20}{11}, -\frac{19}{20}, -\frac{5}{3}, -\frac{25}{44}, -\frac{1}{5}\} = -32 \text{ as } x \geq 0,$$

$$\psi_{99}(x) = \min\{-32, \frac{109}{3}, -\frac{7}{2}, \frac{56}{85}, \frac{7}{4}, \frac{20}{11}, -\frac{19}{20}, -\frac{5}{3}, -\frac{25}{44}+13x, -\frac{1}{5}\} = -32 \text{ as } x \geq 0,$$

and for other i,

$$\psi_i(x) = \min\{-32, \frac{109}{3}, -\frac{7}{2}, \frac{56}{85}, \frac{7}{4}, \frac{20}{11}, -\frac{19}{20}, -\frac{5}{3}, -\frac{25}{44}, -\frac{1}{5}\} = -32 \text{ as } x \geq 0.$$

So

$$\psi_0'(x) = \begin{cases} 18 & \text{, if } 0 \leq x < \frac{19}{12}, \\ 0 & \text{, if } x \geq \frac{19}{12}, \end{cases}$$

$$\psi_1'(x) = \begin{cases} 32 & \text{, if } 0 \leq x < \frac{57}{64}, \\ 0 & \text{, if } x \geq \frac{57}{64}, \end{cases}$$

and $\psi_i'(x) = 0$ as $x \geq 0$, for all $i = 2, \cdots, 100$. By theorem 1.8, we have

$$\begin{aligned} & \lim_{t\to 0} t\frac{d}{dt}\mathcal{M}(t) \\ = \quad & \frac{2}{50}\left(-\frac{32\cdot 49\cdot 101}{100} + \int_0^{\frac{19}{12}} 18\cdot 17dx + \int_0^{\frac{57}{64}} 32\cdot 31dx\right) = -\frac{18859}{1250} < 0. \end{aligned}$$

❑

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